CLASSICAL YANG-BAXTER EQUATION AND THE A_∞-CONSTRAINT

A. POLISHCHUK

Abstract. We show that elliptic solutions of classical Yang-Baxter equation (CYBE) can be obtained from triple Massey products on elliptic curve. We introduce the associative version of this equation which has two spectral parameters and construct its elliptic solutions. We also study some degenerations of these solutions.

Introduction

Recall that the classical Yang-Baxter equation (CYBE) is the equation

$$[r^{12}(x), r^{23}(y)] + [r^{12}(x), r^{13}(x+y)] + [r^{13}(x+y), r^{23}(y)] = 0$$

where $r(x)$ is a meromorphic function of one complex variable x in the neighborhood of 0 taking values in $\mathfrak{g}\otimes\mathfrak{g}$ for some Lie algebra $\mathfrak{g}$. Here $r^{12}(x)$ denotes the element $r(x)\otimes 1 \in U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes U(\mathfrak{g})$, etc. In their remarkable paper [2] Belavin and Drinfeld studied non-degenerate solutions of the CYBE (i.e. solutions such that the tensor $r(x)$ has maximal rank for generic x) for a simple Lie algebra $\mathfrak{g}$. They proved that any such solution is equivalent to either *elliptic, trigonometric, or rational* meaning the character of dependence of $r(x)$ on x. Furthermore, they completely classified elliptic solutions (which can appear only in the case $\mathfrak{g} = \mathrm{sl}_n$) and trigonometric solutions.

In this paper we present an unexpected connection between the CYBE and the A_∞-constraint. The latter is certain generalization of the associativity axiom invented by Stasheff [21]. One can consider the notion of A_∞-algebra (resp. A_∞-category) as a natural replacement for the notion of associative algebra (resp. category) in the presence of a differential. One of the reasons for introducing this notion is that the category of dg-algebras (in which the usual associativity constraint is imposed) doesn't have enough morphisms, so it is often convenient to embed it into the larger category of A_∞-algebras. In this paper we observe that in some special situations triple products in A_∞-category [1] can be arranged into tensors satisfying CYBE. More precisely, we show that all non-degenerate elliptic solutions of the CYBE for sl_n arise in this way from certain triple products in the A_∞-version of the derived category of coherent sheaves on elliptic curve. We also show that all non-degenerate trigonometric solutions of the CYBE for sl_2 arise in the same way from the A_∞-category associated with the union of two $\mathbb{P}^1$'s glued in two points. We expect that one can obtain all non-degenerate trigonometric solutions of the CYBE for sl_n by considering A_∞-categories of singular curves of arithmetic genus 1.

The triple products in A_∞-categories leading to CYBE appear to be specializations of triple products of a more general kind which in turn produce solutions of another equation that we call the *associative Yang-Baxter equation* (AYBE):

$$r^{12}(-u', v)r^{13}(u+u', v+v') - r^{23}(u+u', v')r^{12}(u, v) + r^{13}(u, v+v')r^{23}(u', v') = 0, \qquad (0.1)$$

where $r(u, v)$ is a meromorphic function of two complex variables (u, v) in the neighborhood of $(0, 0)$ taking values in $A\otimes A$ where A is an associative algebra with unit. [2] We conjecture that for $A = \mathrm{Mat}(n, \mathbb{C})$

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[1]One technical detail concerning the above relation between A_∞-constraint and the CYBE is that we need to consider A_∞-structures which have *cyclic symmetry*. This notion is defined in [18] and in [17] we showed that there is a cyclic symmetry on the A_∞-category associated with a complex compact manifold.

[2]Constant solutions of this equation were considered in [1].

the analogue of Belavin-Drinfeld classification holds, i.e. all non-degenerate solutions of the AYBE are equivalent to either elliptic[3], or trigonometric, or rational solutions. In section 4 we check that this is true for scalar solutions, i.e. for $A = \mathbb{C}$. In this case the only solution is the Kronecker's function $F(u, v, \tau)$ (see section 2.2) and its degenerations. The relation between AYBE and CYBE is the following. Let $\mathrm{pr} : \mathrm{Mat}(n, \mathbb{C}) \to \mathrm{sl}_n(\mathbb{C})$ be the projection along scalar matrices. It turns out that in the situations we consider the function $(\mathrm{pr} \otimes \mathrm{pr})(r(u, v))$ has a limit as $u \to 0$. We show that if $r(u, v)$ satisfies the AYBE and the unitarity condition

$$r^{21}(-u, -v) = -r(u, v) \tag{0.2}$$

then the limit $\overline{r}(v) = (\mathrm{pr} \otimes \mathrm{pr})(r(u, v))|_{u=0}$ is a solution of the CYBE. We construct elliptic solutions of the AYBE for $\mathrm{Mat}(n, \mathbb{C})$ which specialize in this way to the usual elliptic r-matrices. Also we construct two trigonometric solutions of the AYBE for $\mathrm{Mat}(2, \mathbb{C})$ which specialize to two different trigonometric solutions of the CYBE for $\mathrm{sl}_2(\mathbb{C})$. In section 5 we show that if $\overline{r}(v)$ is a non-degenerate unitary solution of the CYBE with values in $\mathrm{sl}_n(\mathbb{C})$ which has no infinitesimal symmetries then up to rescaling $r(u, v) \mapsto \exp(cuv)r(u, v)$ (where $c \in \mathbb{C}$) there exists at most one unitary solution of the AYBE with values in $\mathrm{Mat}_n(\mathbb{C})$ of the form $r(u, v) = \frac{1 \otimes 1}{u} + r_0(v) + \dots$ with $(\mathrm{pr} \otimes \mathrm{pr})(r_0(v)) = \overline{r}(v)$. This applies in particular to elliptic r-matrices since they have no infinitesimal symmetries.

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1. Identities between triple Massey products and r-matrices

1.1. **Massey products in A_∞-categories and in triangulated categories.** Recall that an A_∞ category consists of a class of objects, a collection of (graded) vector spaces of morphisms between them equipped with operations $m_n(a_1, \dots, a_n)$ which associate to any sequence $a_1, \dots, a_n$ of composable morphisms ($n \geq 1$) a new morphism (of degree $\sum_i \deg(a_i) + 2 - n$). These operations should satisfy the set of equations similar to the associativity equations which we call A_∞-*constraint*. They have form

$$\sum \pm m_k(a_1, \dots, a_i, m_l(a_{i+1}, \dots, a_{i+l}), \dots, a_n) = 0$$

where $a_1, \dots, a_n$ is a sequence of composable morphisms, the sum is taken over all subsegments in the segment of integers $[1, n]$. The choice of signs is rather subtle (and non-unique). We follow the sign convention of [7]. For more details regarding this definition see [18]. We always impose the condition that our A_∞-category has strict identity morphisms, i.e. m_1-closed elements $\mathrm{id}_X \in \mathrm{Hom}^0(X, X)$ for every object X, which are units with respect to m_2 and such that any higher product m_n ($n \geq 3$) which has id_X as one of the arguments vanishes.

Loosely speaking Massey products in A_∞-categories are expressions in m_n's which are invariant under arbitrary homotopy of A_∞-structure (see [18] for the definition). Unfortunately, the corresponding formalism seems to be absent in the existing literature except in the particular case of a differential graded category which can be considered as an A_∞-category with $m_n = 0$ for $n > 2$.

On the other hand, there is a definition of Massey products in triangulated categories (see [6] IV.2, [16]). These products coincide with the differential graded Massey products in the case when the triangulated category $\mathcal{D}$ is *enhanced* in the sense of Bondal-Kapranov's paper [3]. By definition this means that $\mathcal{D}$ is obtained by taking cohomology of a *pretriangulated dg-category* (the property of a dg-category to be pretriangulated means that certain convolutions exist). Note that according to Kontsevich's philosophy (see [11], [12]) this pretriangulated dg-category should be considered as a primary object (considered up to A_∞-equivalence).

The enhanced triangulated category we are interested in is $D^b(X)$ — the bounded derived category of coherent sheaves on a projective variety X over a field k (see [3]). Let us denote by $D^b_{dg}(X)$ the

[3] Unlike the case of CYBE, "elliptic" here means "elliptic of the third kind", i.e. we allow functions corresponding to meromorphic sections of line bundles on an elliptic curve.

corresponding pretriangulated dg-category. The objects of $D^b_{dg}(X)$ are bounded complexes of coherent sheaves while the morphisms are given by some standard complexes computing the corresponding Ext's. According to general principles of homological perturbation theory (see [10], [8], [9],[13]) there exists an A_∞-category $D^b_\infty(X)$ with the same objects as $D^b_{dg}(X)$ such that $D^b_\infty(X)$ is A_∞-equivalent to $D^b_{dg}(X)$ and $m_1 = 0$ in $D^b_\infty(X)$. Then Massey products in $D^b(X)$ (as in triangulated category) and in D^b_∞ (as in A_∞-category) are the same. The advantage of considering D^b_∞ is that we can apply A_∞-constraint to derive some non-trivial relations between Massey products. On the other hand, Massey products in triangulated categories are easier to compute and they often have a geometric interpretation.

In this paper we will only consider triple Massey products of the particular kind. First, let us recall the definition in the context of triangulated categories. Let X, Y, Z, T be objects of a triangulated category $\mathcal{D}$, $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}^1(Y, Z) := \mathrm{Hom}(Y, Z[1])$, $h \in \mathrm{Hom}(Z, T)$ be morphisms, such that $g \circ f = 0$, $h \circ g = 0$. Then the Massey product

$$MP(f, g, h) \in \mathrm{coker}(\mathrm{Hom}(X, Z) \oplus \mathrm{Hom}(Y, T) \stackrel{(h,f)}{\to} \mathrm{Hom}(X, T))$$

is defined as follows. Let

$$Z \stackrel{\alpha}{\to} C \stackrel{\beta}{\to} Y \stackrel{g}{\to} Z[1] \to \dots$$

be a distinguished triangle. Then by assumption there exist morphisms $\widetilde{f} \in \mathrm{Hom}(X, C)$ and $\widetilde{h} \in \mathrm{Hom}(C, T)$ such that

$$\beta \circ \widetilde{f} = f,$$

$$\widetilde{h} \circ \alpha = h.$$

The Massey product $MP(f, g, h)$ is defined as the class of the element

$$\widetilde{h} \circ \widetilde{f} \in \mathrm{Hom}(X, T).$$

Now let us give a definition of the corresponding triple Massey products in the context of A_∞-categories (see [5]). Let X, Y, Z, T be objects in an A_∞-category $\mathcal{C}$. Let us denote by $H\mathcal{C}$ the graded category obtained from $\mathcal{C}$ by taking cohomologies of Hom with respect to m_1. Then for every triple of morphisms $f \in \mathrm{Hom}^i_{H\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}^j_{H\mathcal{C}}(Y, Z)$, $h \in \mathrm{Hom}^k_{H\mathcal{C}}(Z, T)$ such that $g \circ f = 0$, $h \circ g = 0$ we can define their Massey product

$$MP(f, g, h) \in \mathrm{coker}(\mathrm{Hom}^{i+j-1}_{H\mathcal{C}}(X, Z) \oplus \mathrm{Hom}^{j+k-1}_{H\mathcal{C}}(Y, T) \stackrel{(h,f)}{\to} \mathrm{Hom}^{i+j+k-1}_{H\mathcal{C}}(X, T)).$$

For this we choose m_1-closed elements $\widetilde{f} \in \mathrm{Hom}^i_{\mathcal{C}}(X, Y)$, $\widetilde{g} \in \mathrm{Hom}^j_{\mathcal{C}}(Y, Z)$, $\widetilde{h} \in \mathrm{Hom}^k_{\mathcal{C}}(Z, T)$ representing f, g and h. Furthermore, by assumption we have

$$m_2(\widetilde{f}, \widetilde{g}) = m_1(p),$$

$$m_2(\widetilde{g}, \widetilde{h}) = m_1(q)$$

for some $p \in \mathrm{Hom}^{i+j-1}_{\mathcal{C}}(X, Z)$, $q \in \mathrm{Hom}^{j+k-1}_{\mathcal{C}}(Y, T)$. Then we define $MP(f, g, h)$ as the class of the m_1-closed element

$$m_3(\widetilde{f}, \widetilde{g}, \widetilde{h}) - m_2(p, \widetilde{h}) + (-1)^{\deg f} m_2(\widetilde{f}, q)$$

(the fact that it is m_1-closed follows from the A_∞-constraint). When $m_3 = 0$ this definition coincides with the usual definition given in dg-context. On the other hand, if $m_1 = 0$ then this Massey product coincides with m_3. Finally, we claim that this Massey product is preserved under any equivalence of A_∞-categories. This is a consequence of the following result.

Proposition 1.1. *Let $F : \mathcal{C} \to \mathcal{C}'$ be an A_∞-functor between A_∞-categories, let $HF : H\mathcal{C} \to H\mathcal{C}'$ be the induced functor between the corresponding graded categories. Then*

$$HF(MP(f, g, h) = MP(HF(f), HF(g), HF(h)).$$

Proof. Let $F = (F_n)$ where F_n are maps from n-tuples of composable morphisms in $\mathcal{C}$ to morphisms in $\mathcal{C}'$. According to the definition of the A_∞-functor we have

$$m_2(F_1\widetilde{f}, F_1\widetilde{g}) = F_1 m_2(\widetilde{f}, \widetilde{g}) - m_1 F_2(\widetilde{f}, \widetilde{g}) = F_1 m_1(p) - m_1 F_2(\widetilde{f}, \widetilde{g}).$$

Since F_1 commutes with m_1 we get

$$m_2(F_1\widetilde{f}, F_1\widetilde{g}) = m_1(F_1(p) - F_2(\widetilde{f}, \widetilde{g})).$$

Similarly,

$$m_2(F_1\widetilde{g}, F_1\widetilde{h}) = m_1(F_1(q) - F_2(\widetilde{g}, \widetilde{h})).$$

Thus, the triple Massey product $MP(HF(f), HF(g), HF(h))$ is represented by the element

$$m_3(F_1\widetilde{f}, F_1\widetilde{g}, F_1\widetilde{h}) - m_2(F_1(p) - F_2(\widetilde{f}, \widetilde{g}), F_1\widetilde{h}) + (-1)^{\deg f} m_2(F_1\widetilde{f}, F_1(q) - F_2(\widetilde{g}, \widetilde{h})).$$

Using the identity

$$m_3(F_1\widetilde{f}, F_1\widetilde{g}, F_1\widetilde{h}) + m_2(F_2(\widetilde{f}, \widetilde{g}), F_1\widetilde{h}) - (-1)^{\deg f} m_2(F_1\widetilde{f}, F_2(\widetilde{g}, \widetilde{h})) =$$
$$F_1 m_3(\widetilde{f}, \widetilde{g}, \widetilde{h}) - F_2(m_2(\widetilde{f}, \widetilde{g}), \widetilde{h}) - (-1)^{\deg f} F_2(\widetilde{f}, m_2(\widetilde{g}, \widetilde{h})) - m_1 F_3(\widetilde{f}, \widetilde{g}, \widetilde{h}).$$

we can rewrite the element representing $MP(HF(f), HF(g), HF(h))$ as follows:

$$\begin{aligned} &F_1 m_3(\widetilde{f}, \widetilde{g}, \widetilde{h}) - m_2(F_1(p), F_1\widetilde{h}) + (-1)^{\deg f} m_2(F_1\widetilde{f}, F_1(q)) \\ &-F_2(m_1(p), \widetilde{h}) - (-1)^{\deg f} F_2(\widetilde{f}, m_1(q)) - m_1 F_3(\widetilde{f}, \widetilde{g}, \widetilde{h}). \end{aligned} \tag{1.1}$$

Note that the last term is a coboundary, hence, it can be omitted. On the other hand, we have

$$m_2(F_1 p, F_1\widetilde{h}) \equiv F_1 m_2(p, \widetilde{h}) - F_2(m_1(p), \widetilde{h}) \mod \operatorname{Im}(m_1)$$

and

$$m_2(F_1\widetilde{f}, F_1 q) \equiv F_1 m_2(\widetilde{f}, q) + F_2(\widetilde{f}, m_1(q)) \mod \operatorname{Im}(m_1).$$

Substituting this in (1.1) we obtain that $MP(HF(f), HF(g), HF(h))$ is represented by

$$F_1 m_3(\widetilde{f}, \widetilde{g}, \widetilde{h}) - F_1 m_2(p, \widetilde{h}) + (-1)^{\deg f} F_1 m_2(\widetilde{f}, q).$$

Therefore, it coincides with $HF(MP(f, g, h))$. □

Both the definitions above can be slightly generalized: instead of considering a decomposable tensor $f \otimes g \otimes h$ one can take any tensor in the appropriate subspace of $\operatorname{Hom}^i(X, Y) \otimes \operatorname{Hom}^j(Y, Z) \otimes \operatorname{Hom}^k(Z, T)$. We leave this to the reader (in the context of triangulated categories the corresponding definition can be found in [16]).

1.2. **Generic identity and the associative Yang-Baxter equation.** Let $\mathcal{C}$ be an A_∞-category with $m_1 = 0$. Assume that we have two families $\mathcal{M}$ and $\mathcal{M}'$ of objects of $\mathcal{C}$ with the following properties:
(i) for every pair of distinct objects $X_1, X_2 \in \mathcal{M}$ (resp. $Y_1, Y_2 \in \mathcal{M}'$) one has $\operatorname{Hom}^\bullet(X_1, X_2) = 0$ (resp. $\operatorname{Hom}^\bullet(Y_1, Y_2) = 0$);
(ii) for every $X \in \mathcal{M}$ and every $Y \in \mathcal{M}'$ the space $\operatorname{Hom}^\bullet(X, Y)$ is concentrated in degree 0, the space $\operatorname{Hom}^\bullet(Y, X)$ is concentrated in degree 1 and a perfect pairing

$$\langle \cdot, \cdot \rangle : \operatorname{Hom}^0(X, Y) \otimes \operatorname{Hom}^1(Y, X) \to k$$

is given.

In this situation we can consider the triple products

$$m_3 : \operatorname{Hom}^0(X_1, Y_1) \otimes \operatorname{Hom}^1(Y_1, X_2) \otimes \operatorname{Hom}^0(X_2, Y_2) \to \operatorname{Hom}^0(X_1, Y_2)$$

and

$$m_3 : \operatorname{Hom}^1(Y_1, X_2) \otimes \operatorname{Hom}^0(X_2, Y_2) \otimes \operatorname{Hom}^1(Y_2, X_1) \to \operatorname{Hom}^1(Y_1, X_1)$$

where $X_1, X_2 \in \mathcal{M}$, $X_1 \neq X_2$, $Y_1, Y_2 \in \mathcal{M}'$, $Y_1 \neq Y_2$. Using the vanishing of the spaces $\operatorname{Hom}^\bullet(X_1, X_2)$ and $\operatorname{Hom}^\bullet(Y_1, Y_2)$ and the condition $m_1 = 0$ one can immediately see that the corresponding Massey

products coincide with m_3. We assume in addition that the pairing from (ii) is compatible with these triple products in the following sense:
(iii) for every $f_1 \in \mathrm{Hom}^0(X_1, Y_1)$, $g_1 \in \mathrm{Hom}^1(Y_1, X_2)$, $f_2 \in \mathrm{Hom}^0(X_2, Y_2)$, $g_2 \in \mathrm{Hom}^1(Y_2, X_1)$ one has

$$\langle m_3(f_1, g_1, f_2), g_2\rangle = -\langle f_1, m_3(g_1, f_2, g_2)\rangle = -\langle m_3(f_2, g_2, f_1), g_1\rangle.$$

Note that the condition (iii) is satisfied when $\mathcal{C}$ has a structure of *cyclic* A_∞-category in the sense of [17].

Using the duality from (ii) we can rewrite the tensor corresponding to m_3 as a linear map

$$r^{X_1X_2}_{Y_1Y_2} : \mathrm{Hom}^0(X_1, Y_1) \otimes \mathrm{Hom}^0(X_2, Y_2) \to \mathrm{Hom}^0(X_2, Y_1) \otimes \mathrm{Hom}^0(X_1, Y_2).$$

Theorem 1. *For any triples of distinct objects $X_1, X_2, X_3 \in \mathcal{M}$, $Y_1, Y_2, Y_3 \in \mathcal{M}'$ one has*

$$(r^{X_3X_2}_{Y_1Y_2})^{12}(r^{X_1X_3}_{Y_1Y_3})^{13} - (r^{X_1X_3}_{Y_2Y_3})^{23}(r^{X_1X_2}_{Y_1Y_2})^{12} + (r^{X_1X_2}_{Y_1Y_3})^{13}(r^{X_2X_3}_{Y_2Y_3})^{23} = 0 \tag{1.2}$$

as a map

$$\mathrm{Hom}^0(X_1, Y_1)\,\mathrm{Hom}^0(X_2, Y_2)\,\mathrm{Hom}^0(X_3, Y_3) \to \mathrm{Hom}^0(X_2, Y_1)\,\mathrm{Hom}^0(X_3, Y_2)\,\mathrm{Hom}^0(X_1, Y_3).$$

In addition the following skew-symmetry holds:

$$(r^{X_1X_2}_{Y_1Y_2})^{21} = -r^{X_2X_1}_{Y_2Y_1}. \tag{1.3}$$

Proof. The skew-symmetry follows easily from the property (iii). Using it we can rewrite the equation (1.2) as follows

$$(r^{X_1X_3}_{Y_2Y_3})^{23}(r^{X_1X_2}_{Y_1Y_2})^{12} + c.p. = 0$$

where "c.p." stands for the terms obtained from the first one by cyclic permutation of indices.

Let us consider any six elements $f_i \in \mathrm{Hom}^0(X_i, Y_i)$, $g_i \in \mathrm{Hom}^1(Y_i, X_{i+1})$, where $i \in \mathbb{Z}/3\mathbb{Z}$ (so that $X_4 := X_1$). The definition of $r^{X_1X_2}_{Y_1Y_2}$ is equivalent to the following formula:

$$\langle r^{X_1X_2}_{Y_1Y_2}(f_1 \otimes f_2), g_1\rangle_1 = m_3(f_1, g_1, f_2),$$

where $\langle ?, ?\rangle_1$ denotes the result of applying the pairing $\langle ?, ?\rangle$ in the first component of the tensor product. It follows that

$$\langle (r^{X_1X_3}_{Y_2Y_3})^{23}(r^{X_1X_2}_{Y_1Y_2})^{12}(f_1 \otimes f_2 \otimes f_3), g_1 \otimes g_2\rangle_{12} = m_3(m_3(f_1, g_1, f_2), g_2, f_3)$$

where $\langle ?, ?\rangle_{12}$ denotes the pairing $\langle ?, ?\rangle$ applied in the first two components of the tensor product. Thus, we have

$$\langle (r^{X_1X_3}_{Y_2Y_3})^{23}(r^{X_1X_2}_{Y_1Y_2})^{12}(f_1 \otimes f_2 \otimes f_3), g_1 \otimes g_2 \otimes g_3\rangle = \langle m_3(m_3(f_1, g_1, f_2), g_2, f_3), g_3\rangle.$$

Using property (iii) we can rewrite this formula as follows:

$$\langle (r^{X_1X_3}_{Y_2Y_3})^{23}(r^{X_1X_2}_{Y_1Y_2})^{12}(f_1 \otimes f_2 \otimes f_3), g_1 \otimes g_2 \otimes g_3\rangle = -\langle m_3(f_1, g_1, f_2), m_3(g_2, f_3, g_3)\rangle. \tag{1.4}$$

On the other hand, applying the A_∞-constraint to five composable morphisms f_1, g_1, f_2, g_2, f_3 and using property (i) we get

$$m_3(m_3(f_1, g_1, f_2), g_2, f_3) + m_3(f_1, m_3(g_1, f_2, g_2), f_3) - m_3(f_1, g_1, m_3(f_2, g_2, f_3)) = 0. \tag{1.5}$$

Pairing this identity with g_3 and using property (iii) we get

$$\begin{gathered}\langle m_3(f_1, g_1, f_2), m_3(g_2, f_3, g_3)\rangle + \langle m_3(f_3, g_3, f_1), m_3(g_1, f_2, g_2)\rangle + \\ \langle m_3(f_2, g_2, f_3), m_3(g_3, f_1, g_1)\rangle = 0.\end{gathered} \tag{1.6}$$

□

Let A be an associative k-algebra with a unit. For a tensor $r^{X_1X_2}_{Y_1Y_2} \in A \otimes_k A$ depending on two sets of variables $X_1, X_2 \in \mathcal{M}$, $Y_1, Y_2 \in \mathcal{M}'$ the equation (1.2) can be considered as an associative version of

the classical Yang-Baxter equation. In the case when there is no dependence on variables we obtain the equation

$$r^{12}r^{13} - r^{23}r^{12} + r^{13}r^{23} = 0$$

which was considered in [1] in connection with infinitesimal Hopf algebras.

Now let $k = \mathbb{C}$. Similar to the case of the usual classical Yang-Baxter equation it is natural to consider solutions with complex variables X_i, Y_j such that $r = r(u, v)$ is a meromorphic function of $u = X_1 - X_2$ and $v = Y_1 - Y_2$ (where u and v vary in the neighborhood of 0). Then the equation can be rewritten in the form (0.1) while the skew-symmetry equation becomes the equation (0.2). Using the above theorem we will construct below elliptic solutions of the AYBE satisying the condition (0.2) with values in the matrix algebra $\mathrm{Mat}(n, \mathbb{C})$ which specialize to the standard elliptic r-matrices for $\mathrm{sl}_n(\mathbb{C})$ as u tends to 0. This limit procedure works more generally as follows. We say that a solution $r(u, v)$ of the AYBE is *unitary* if it satisfies the equation (0.2). Similarly, a unitary solution of the CYBE is a solution satisfying the equation $\overline{r}^{21}(-v) = -\overline{r}(v)$.

Lemma 1.2. *Let $r(u, v)$ be a unitary solution of the AYBE with values in $\mathrm{Mat}(n, \mathbb{C})$. Let $\mathrm{pr} : \mathrm{Mat}(n, \mathbb{C}) \to \mathrm{sl}_n(\mathbb{C})$ be the projection along scalar matrices. Assume that $(\mathrm{pr} \otimes \mathrm{pr})(r(u, v))$ has a limit as $u \to 0$. Then $\overline{r}(v) = (\mathrm{pr} \otimes \mathrm{pr})(r(u, v))|_{u=0}$ is a unitary solution of the CYBE.*

Proof. Applying the permutation of the first two factors to the equation (0.1) and making a change of variables $(v, v') \mapsto (-v, v + v')$, $(u, u') \mapsto (u', u)$ we obtain

$$r^{21}(-u, -v)r^{23}(u + u', v') - r^{13}(u + u', v + v')r^{21}(u', -v) + r^{23}(u', v')r^{13}(u, v + v') = 0.$$

Using the equation (0.2) this equation can be rewritten as follows:

$$-r^{12}(u, v)r^{23}(u + u', v') + r^{13}(u + u', v + v')r^{12}(-u', v) + r^{23}(u', v')r^{13}(u, v + v') = 0.$$

Subtracting this equation from (0.1) we get

$$[r^{12}(-u', v), r^{13}(u + u', v + v')] - [r^{23}(u + u', v'), r^{12}(u, v)] + [r^{13}(u, v + v'), r^{23}(u', v')] = 0.$$

Finally, applying $\mathrm{pr} \otimes \mathrm{pr}$ and substituting $u = u' = 0$ we obtain that $\overline{r}(v)$ satisfies CYBE. $\square$

There is a natural notion of equivalence for the solutions of (1.2). Namely, if φ^X_Y is a function with values in A^* (invertible elements in A) and $r^{X_1X_2}_{Y_1Y_2}$ is a solution of (1.2) then

$$\widetilde{r}^{X_1X_2}_{Y_1Y_2} = (\varphi^{X_2}_{Y_1} \otimes \varphi^{X_1}_{Y_2})r^{X_1X_2}_{Y_1Y_2}(\varphi^{X_1}_{Y_1} \otimes \varphi^{X_2}_{Y_2})^{-1}$$

is also a solution of (1.2). We will call the solutions $\widetilde{r}$ and r *equivalent*. On the other hand, if ψ_Y is a function with values in $\mathrm{Aut}(A)$ then we can construct a new solution by looking at

$$(\psi_{Y_1} \otimes \psi_{Y_2})r^{X_1X_2}_{Y_1Y_2}.$$

However, in the case of the matrix algebra this doesn't give anything new since all automorphisms are inner.

It is easy to see that if $r(u, v)$ is a solution of (0.1) then

$$c_1 \cdot \exp(c_2uv) \cdot r(u, v)$$

is also a solution for arbitrary constants $c_1 \in \mathbb{C}^*$ and $c_2 \in \mathbb{C}$. We will call this operation *rescaling* of a solution.

It seems reasonable to conjecture that all unitary solutions of (0.1) with values in the matrix algebra satisfying the non-degeneracy condition (that the tensor $r(u, v)$ is non-degenerate for generic u, v) are equivalent (up to rescaling) to either *elliptic* or *trigonometric* or *rational* solution similar to the Belavin-Drinfeld classification in [2]. In section 4 we will check our conjecture in the simplest case $n = 1$, i.e. we will classify scalar unitary solutions of (0.1).

1.3. **Classical Yang-Baxter equation.** Now we will express the "limit" of $r^{X_1 X_2}_{Y_1 Y_2}$ as X_2 tends to X_1 directly in terms of A_∞-structure. We will see that in the case $X_1 = X_2$ the Massey products have smaller domain of definition and smaller range and that the corresponding tensor satisfies the CYBE.

We still consider an A_∞-category $\mathcal{C}$ with $m_1 = 0$. Now assume that we have an object X and a family of objects $\mathcal{M}$ in $\mathcal{C}$, such that the following properties hold:
(i)' For every pair of distinct objects $Y_1, Y_2 \in \mathcal{M}$ one has $\mathrm{Hom}^\bullet(Y_1, Y_2) = 0$; the spaces $\mathrm{Hom}^0(X, X)$ and $\mathrm{Hom}^1(X, X)$ are one-dimensional, $\mathrm{Hom}^i(X, X) = 0$ for $i \neq 0, 1$.
(ii)' for every $Y \in \mathcal{M}$ the space $\mathrm{Hom}^\bullet(X, Y)$ is concentrated in degree 0, the space $\mathrm{Hom}^\bullet(Y, X)$ is concentrated in degree 1 and the composition map

$$m_2 : \mathrm{Hom}^0(X, Y) \otimes \mathrm{Hom}^1(Y, X) \to \mathrm{Hom}^1(X, X) \simeq k$$

is a perfect pairing.

In this situation we can consider the Massey product induced by the triple product

$$m_3 : \mathrm{Hom}^0(X, Y_1) \otimes \mathrm{Hom}^1(Y_1, X) \otimes \mathrm{Hom}^0(X, Y_2) \to \mathrm{Hom}^0(X, Y_2) \tag{1.7}$$

where $Y_1, Y_2 \in \mathcal{M}$, $Y_1 \neq Y_2$. The domain of definition of the corresponding triple Massey product contains tensors $\sum_i f_i \otimes g_i \otimes h$ such that

$$\sum_i m_2(f_i, g_i) = 0.$$

The value of the Massey product on such a tensor is an element of $\mathrm{Hom}^0(X, Y_2)$ defined up to addition of a scalar multiple of h. It is more convenient to consider the product (1.7) as a linear map

$$\mathrm{Hom}^0(X, Y_1) \otimes \mathrm{Hom}^1(Y_1, X) \to \mathrm{End}(\mathrm{Hom}^0(X, Y_2)).$$

Then the corresponding Massey product is the map

$$K_{X,Y_1} \to \mathrm{End}(\mathrm{Hom}^0(X, Y_2))/k \cdot \mathrm{id}, \tag{1.8}$$

where $K_{X,Y_i} \subset \mathrm{Hom}^0(X, Y_i)\,\mathrm{Hom}^1(Y_i, X)$ is the kernel of m_2.

For every finite-dimensional vector space V over k let us denote by $\mathrm{sl}(V) \subset \mathrm{End}(V)$ the subspace of traceless endomorphisms, and $\mathrm{pgl}(V) = \mathrm{End}(V)/k \cdot \mathrm{id}$. We have a canonical isomorphism $\mathrm{sl}(V)^* \simeq \mathrm{pgl}(V)$ induced by self-duality of $\mathrm{End}(V)$.

Let us choose a linear isomorphism $\mathrm{tr} : \mathrm{Hom}^1(X, X) \to k$. Then using the pairing

$$\langle \cdot, \cdot \rangle = \mathrm{tr} \circ m_2 : \mathrm{Hom}^0(X, Y_i) \otimes \mathrm{Hom}^1(Y_i, X) \to k$$

we can identify $\mathrm{Hom}^1(Y_i, X)$ with the dual space to $\mathrm{Hom}^0(X, Y_i)$. In view of this duality the triple product (1.7) can be considered as a tensor

$$\widetilde{r}_{Y_1,Y_2} \in \mathrm{End}(\mathrm{Hom}^0(X, Y_1)) \otimes \mathrm{End}(\mathrm{Hom}^0(X, Y_2)).$$

On the other hand, K_{X,Y_1} can be identified with the subspace $\mathrm{sl}(\mathrm{Hom}^0(X, Y_1)) \subset \mathrm{End}(\mathrm{Hom}^0(X, Y_1))$. Thus, we can rewrite the map (1.8) as a linear map

$$\mathrm{sl}(\mathrm{Hom}^0(X, Y_1)) \to \mathrm{pgl}(\mathrm{Hom}^0(X, Y_2))$$

or equivalently as a tensor

$$r_{Y_1,Y_2} = r^X_{Y_1,Y_2} \in \mathrm{pgl}(\mathrm{Hom}^0(X, Y_1)) \otimes \mathrm{pgl}(\mathrm{Hom}^0(X, Y_2)).$$

It is easy to see that r_{Y_1,Y_2} is the image of $\widetilde{r}_{Y_1,Y_2}$ under the natural projection. By Proposition 1.1 the tensor r_{Y_1,Y_2} is invariant under any homotopy of A_∞-structure.

We assume in addition that
(iii)' for every $f_i \in \mathrm{Hom}^0(X, Y_i)$, $g_i \in \mathrm{Hom}^1(Y_i, X)$, $i = 1, 2$, one has

$$\langle m_3(f_1, g_1, f_2), g_2 \rangle = -\langle f_1, m_3(g_1, f_2, g_2) \rangle = -\langle m_3(f_2, g_2, f_1), g_1 \rangle.$$

Theorem 2. *For every triple of distinct objects $Y_1, Y_2, Y_3 \in \mathcal{M}$ one has*

$$[r^{12}_{Y_1,Y_2}, r^{13}_{Y_1,Y_3}] + [r^{12}_{Y_1,Y_2}, r^{23}_{Y_2,Y_3}] + [r^{13}_{Y_1,Y_3}, r^{23}_{Y_2,Y_3}] = 0 \tag{1.9}$$

in the Lie algebra $\mathrm{pgl}(\mathrm{Hom}^0(X,Y_1)) \otimes \mathrm{pgl}(\mathrm{Hom}^0(X,Y_2)) \otimes \mathrm{pgl}(\mathrm{Hom}^0(X,Y_3))$*. In addition the following skew-symmetry holds:*

$$r^{21}_{Y_1,Y_2} = -r_{Y_2,Y_1}. \tag{1.10}$$

Proof. Let us consider six elements $f_i \in \mathrm{Hom}^0(X,Y_i)$, $g_i \in \mathrm{Hom}^1(Y_i,X)$, where $i \in \mathbb{Z}/3\mathbb{Z}$, such that $\langle f_i, g_i\rangle = 0$ for all i. In fact, the argument below should (and can) be applied to a slightly more general data: each tensor $f_i \otimes g_i$ should be replaced by an arbitrary element of K_{X,Y_i}. However, we restrict ourself to the case of decomposable tensors to simplify notations. By definition we have

$$\langle \widetilde{r}_{Y_1,Y_2}(f_1 \otimes f_2), g_1 \otimes g_2\rangle = \langle m_3(f_1,g_1,f_2), g_2\rangle.$$

Together with the property (iii)' this immediately implies the skew-symmetry of r. Using it we can rewrite the equation (1.9) in the following form:

$$[r^{12}_{Y_1,Y_2}, r^{23}_{Y_2,Y_3}] + c.p. = 0.$$

It is easy to see that

$$\begin{aligned}&\langle \widetilde{r}^{23}_{Y_2,Y_3}\widetilde{r}^{12}_{Y_1,Y_2}(f_1\otimes f_2\otimes f_3), g_1\otimes g_2\otimes g_3\rangle = \langle m_3(m_3(f_1,g_1,f_2),g_2,f_3),g_3\rangle = \\ &-\langle m_3(f_1,g_1,f_2), m_3(g_2,f_3,g_3)\rangle.\end{aligned}$$

The A_∞-constraint applied to the morphisms f_1, g_1, f_2, g_2, f_3 differs from (1.5) by one additional term:

$$\begin{aligned}&m_3(m_3(f_1,g_1,f_2),g_2,f_3) + m_3(f_1, m_3(g_1,f_2,g_2),f_3) - m_3(f_1,g_1,m_3(f_2,g_2,f_3)) - \\ &m_2(m_4(f_1,g_1,f_2,g_2),f_3) = 0.\end{aligned}$$

However, this additional term drops out when we apply pairing with g_3 since $m_4(f_1,g_1,f_2,g_2)$ is a multiple of id_X and $\langle f_3, g_3\rangle = 0$. Thus, the equality (1.6) still holds in our situation. It follows that the tensor

$$\widetilde{r}^{23}_{Y_2,Y_3}\widetilde{r}^{12}_{Y_1,Y_2} + c.p. \in \mathrm{End}(\mathrm{Hom}^0(X,Y_1)\otimes \mathrm{Hom}^0(X,Y_2)\otimes \mathrm{Hom}^0(X,Y_3))$$

is orthogonal to $\mathrm{sl}(\mathrm{Hom}^0(X,Y_1))\otimes \mathrm{sl}(\mathrm{Hom}^0(X,Y_2))\otimes \mathrm{sl}(\mathrm{Hom}^0(X,Y_3))$. Hence, its projection to

$$\mathrm{pgl}(\mathrm{Hom}^0(X,Y_1))\otimes \mathrm{pgl}(\mathrm{Hom}^0(X,Y_2)\otimes \mathrm{pgl}(\mathrm{Hom}^0(X,Y_3))$$

is zero. Similar statement holds for the tensor $\widetilde{r}^{12}\widetilde{r}^{23} + c.p.$ so we are done. □

Assuming in addition that all the spaces $\mathrm{Hom}^0(X,Y)$ for $Y \in \mathcal{M}$ have the same dimension n (this is true in all examples) we can choose isomorphisms $\mathrm{Hom}^0(X,Y) \simeq k^n$ and consider r_{Y_1,Y_2} as an element of $\mathrm{pgl}_n \otimes \mathrm{pgl}_n$. Then the map $(Y_1,Y_2) \mapsto r_{Y_1,Y_2}$ defined on all pairs such that $Y_1 \not\simeq Y_2$ is a solution of the CYBE for pgl_n. A different choice of isomorphisms $\mathrm{Hom}^0(X,Y) \simeq k^n$ leads to an equivalent solution. In the case $k = \mathbb{C}$ one often has a situation when objects X_i and Y_j are parametrized by complex variables and all the spaces $\mathrm{Hom}(X_i,Y_j)$ can be identified with $\mathbb{C}^n$ in such a way that tensors $r^{X_1,X_2}_{Y_1,Y_2}$ (resp. $r^X_{Y_1,Y_2}$) depend only on differences of complex parameters corresponding to X_1, X_2 and Y_1, Y_2. In this case the solutions of the CYBE corresponding to $r^X_{Y_1,Y_2}$ are obtained from the solutions of the AYBE corresponding to $r^{X_1,X_2}_{Y_1,Y_2}$ by the limit procedure described in lemma 1.2.

The above proof also shows that the tensor $r_{Y_1,Y_2} \in \mathrm{pgl}_n \otimes \mathrm{pgl}_n$ has the following property in addition to the CYBE: there exists a lifting $\widetilde{r}_{Y_1,Y_2} \in \mathrm{gl}_n \otimes \mathrm{gl}_n$ of r_{Y_1,Y_2} such that

$$\widetilde{r}^{23}_{Y_2,Y_3}\widetilde{r}^{12}_{Y_1,Y_2} + c.p.$$

projects to zero in $\mathrm{pgl}_n^{\otimes 3}$. It would be interesting to study which solutions of the CYBE satisfy this property.

1.4. **Spherical objects.** Let $\mathcal{D}$ be a triangulated category over a field k, such that all spaces $\mathrm{Hom}(X,Y)$ are finite-dimensional. We use the notation $\mathrm{Hom}^i(X,Y) := \mathrm{Hom}(X, Y[i])$.

Following [20] we call an object $F \in \mathcal{D}$ *n-spherical* if $\mathrm{Hom}^i(F,F) = 0$ for $i \neq 0, n$, $\mathrm{Hom}^0(F,F) \simeq \mathrm{Hom}^n(F,F) \simeq k$, and for every $X \in \mathcal{D}$ the composition map

$$\mathrm{Hom}^i(F,X)\,\mathrm{Hom}^{n-i}(X,F) \to \mathrm{Hom}^n(F,F) \simeq k$$

is a perfect pairing.

In the case when $\mathcal{D}$ is enhanced in the sense of [3] one can define the autoequivalence $T_F : \mathcal{D} \to \mathcal{D}$ such that for every object $X \in \mathcal{D}$ with $\mathrm{Hom}^i(F,X) = 0$ for $i \neq 0$ there is an exact triangle

$$\mathrm{Hom}^0(F,X) \otimes F \to X \to T_F X \to \dots$$

The case when $\mathcal{D}$ is a subcategory in the bounded derived category of quasicoherent sheaves on a projective variety was considered in details by Seidel and Thomas in [20]. The general case of an enhanced triangulated category is similar. It seems that the construction of the functor T_F can be generalized to the case when $\mathcal{D}$ has a structure of triangulated A_∞-category as defined by Kontsevich [12].

It is easy to see that all spherical objects in the derived category of coherent sheaves on an elliptic curve E are (up to shift) either simple vector bundles or structure sheaves of points. In particular, we observe that the group of autoequivalences of $D^b(E)$ acts transitively on the set of isomorphism classes of spherical objects. It seems to be an interesting problem to classify spherical objects in the case when E is replaced by a singular projective curve of arithmetic genus 1. It is natural to consider only such curves for which the structure sheaf $\mathcal{O}$ coincides with the dualizing sheaf. In this case $\mathcal{O}$ and structure sheaves of smooth points are spherical. The corresponding functor $T_{\mathcal{O}}$ together with tensorings by line bundles and automorphisms of the curve generate a large group of autoequivalences of the derived category. In particular, we obtain a lot of spherical objects. However, it is not known whether in this case the group of autoequivalences acts transitively on spherical objects.

1.5. **Non-degeneracy criterion.** From now on we will always work in an enhanced triangulated category which has a cyclic symmetry considered as an A_∞-category. We also keep the notations of sections 1.3 and 1.4. Recall that a tensor $t \in V_1 \otimes V_2$ is called *non-degenerate* if it induces an isomorphism $V_1^\vee \to V_2$. We define the non-degeneracy condition for the tensor $r^{X_1X_2}_{Y_1Y_2}$ by considering it as an element of

$$(\mathrm{Hom}^0(X_1,Y_1)^\vee \otimes \mathrm{Hom}^0(X_2,Y_1)) \otimes (\mathrm{Hom}^0(X_2,Y_2)^\vee \otimes \mathrm{Hom}^0(X_1,Y_2)).$$

Theorem 3. *Assume that Y_1 and Y_2 are 1-spherical. Then the tensor $r^{X_1X_2}_{Y_1Y_2}$ (resp. $r^X_{Y_1,Y_2}$) is non-degenerate if and only if $\mathrm{Hom}^i(T_{Y_2}X_1, T_{Y_1}X_2) = 0$ (resp. $\mathrm{Hom}^i(T_{Y_2}X, T_{Y_1}X) = 0$) for $i = 1,2$.*

Proof. Let us first consider the tensor $r^X_{Y_1,Y_2}$. Using the definition of the Massey product in the context of triangulated categories (see section 1.1) we obtain that $r^X_{Y_1,Y_2}$ corresponds to the composition map

$$\mathrm{Hom}^0(X, T_{Y_1}X) \otimes \mathrm{Hom}^0(T_{Y_1}X, Y_2) \to \mathrm{Hom}^0(X, Y_2). \tag{1.11}$$

More precisely, the exact triangle

$$X \to T_{Y_1}X \to \mathrm{Hom}^1(Y_1,X) \otimes Y_1 \to \dots$$

induces the exact sequence

$$0 \to \mathrm{Hom}^0(X,X) \to \mathrm{Hom}^0(X, T_{Y_1}X) \to K_{X,Y_1} \to 0$$

and an isomorphism

$$\mathrm{Hom}^0(T_{Y_1}X, Y_2) \widetilde{\to} \mathrm{Hom}^0(X, Y_2).$$

Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}^0(X, T_{Y_1}X) & \xrightarrow{\alpha} \mathrm{Hom}^0(T_{Y_1}X, Y_2)^{\vee}\,\mathrm{Hom}^0(X, Y_2) \simeq & \mathrm{Hom}^0(X, Y_2)^{\vee}\,\mathrm{Hom}^0(X, Y_2) \\ \downarrow & & \downarrow \\ K_{X,Y_1} & \xrightarrow{r^X_{Y_1,Y_2}} & \mathrm{pgl}(\mathrm{Hom}^0(X, Y_2)) \end{array} \tag{1.12}$$

where the map α is obtained from (1.11) by dualization. By definition the map α sends the one-dimensional subspace $\mathrm{Hom}^0(X, X) \subset \mathrm{Hom}^0(X, T_{Y_1}X)$ to the span of the identity in $\mathrm{End}(\mathrm{Hom}^0(X, Y_2))$. Thus, the tensor $r^X_{Y_1Y_2}$ is non-degenerate if and only if α is an isomorphism. To this end we observe that α is obtained by applying the functor $\mathrm{Hom}^0(X, ?)$ to the second arrow of the following exact triangle:

$$T^{-1}_{Y_2}T_{Y_1}X \to T_{Y_1}X \to \mathrm{Hom}^0(T_{Y_1}X, Y_2)^{\vee} \otimes Y_2 \to \ldots$$

If $\mathrm{Hom}^i(X, T^{-1}_{Y_2}T_{Y_1}X) = 0$ for $i = 0, 1$ then clearly, α is an isomorphism. To show that the converse is true we have to check that $\mathrm{Hom}^{-1}(X, Y_2) = 0$ and $\mathrm{Hom}^1(X, T_{Y_1}X) = 0$. The first vanishing holds by the assumption (ii)'. From the exact triangle defining $T_{Y_1}X$ we obtain the following long exact sequence:

$$\mathrm{Hom}^0(X, Y_1)\,\mathrm{Hom}^1(Y_1, X) \to \mathrm{Hom}^1(X, X) \to \mathrm{Hom}^1(X, T_{Y_1}X) \to \mathrm{Hom}^1(X, Y_1)\,\mathrm{Hom}^1(Y_1, X) \to \ldots$$

Now the condition (ii)' implies that the first arrow is surjective and the last term vanishes, hence, $\mathrm{Hom}^1(X, T_{Y_1}X) = 0$.

In the case of the tensor $r^{X_1X_2}_{Y_1Y_2}$ the proof is very similar (but more simple): one has natural isomorphisms

$$\mathrm{Hom}^0(X_1, T_{Y_1}X_2) \simeq \mathrm{Hom}^0(X_1, Y_1) \otimes \mathrm{Hom}^1(Y_1, X_2),$$

$$\mathrm{Hom}^0(T_{Y_1}X_2, Y_2) \simeq \mathrm{Hom}^0(X_2, Y_2),$$

while the corresponding Massey product is given by a composition

$$\mathrm{Hom}^0(X_1, T_{Y_1}X_2) \otimes \mathrm{Hom}^0(T_{Y_1}X_2, Y_2) \to \mathrm{Hom}^0(X_1, Y_2)$$

Thus, the non-degeneracy is equivalent to the condition that the map

$$\mathrm{Hom}^0(X_1, T_{Y_1}X_2) \to \mathrm{Hom}^0(T_{Y_1}X_2, Y_2)^{\vee} \otimes \mathrm{Hom}^0(X_1, Y_2)$$

is an isomorphism. Now the proof can be completed similar to the case of $r^X_{Y_1Y_2}$. □

1.6. **Solutions associated with simple vector bundles.** Now let us consider a more specific situation in which the general categorical setup described above is realized. Namely as an enhanced triangulated category we will take the derived category of a projective curve C of arithmetic genus 1. The objects X_i will be simple vector bundles while the objects Y_i will be structure sheaves of smooth points. For simplicity let us assume that C is reduced and it is either irreducible or it is a union of $\mathbb{P}^1$'s intersecting transversally. Then the dualizing sheaf of C is $\mathcal{O}_C$ which implies that most of the conditions (i)-(iii) (resp. (i)'-(iii)') are satisfied automatically. More precisely, to check them one can use the following two lemmas (which are easy consequences of Riemann-Roch theorem and Serre duality on the curve C).

Lemma 1.3. *Let V be a vector bundle on C. Then $\chi(C, V) = \deg V$ where $\deg(V)$ is the sum of degrees of restrictions of V to irreducible components of C.*

Lemma 1.4. *Let X be a simple vector bundle on C or a structure sheaf of a smooth point on C. Then $\mathrm{Ext}^i(X, X) = 0$ for $i \neq 0, 1$, $\mathrm{Ext}^1(X, X) \simeq k$ and the pairing*

$$\mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, X[1]) \to \mathrm{Ext}^1(X, X) \simeq k$$

is non-degenerate for any object Y of the bounded derived category of coherent sheaves on C.

The only remaining condition to be checked is that all Hom^0 and Ext^1 between two simple bundles in question vanish. For example, this is true when these bundles are of the form $(V, V \otimes \mathcal{L})$ where $\mathcal{L}$ is a line bundle on C which has degree zero and is not annihilated by $\mathrm{rk}\, V$ in $\mathrm{Pic}(C)$. The corresponding triple Massey products are computed in the following theorem.

Theorem 4. *(a) Let V_1, V_2 be a pair of simple bundles on C such that $\mathrm{Hom}^0(V_1, V_2) = \mathrm{Ext}^1(V_1, V_2) = 0$. Let y_1, y_2 be a pair of distinct smooth points of C. Then the tensor*

$$r^{V_1,V_2}_{\mathcal{O}_{y_1},\mathcal{O}_{y_2}} \in V_{1,y_1} \otimes V^{\vee}_{2,y_1} \otimes V^{\vee}_{1,y_2} \otimes V_{2,y_2}$$

corresponds to the following composition

$$\mathrm{Hom}(V_{1,y_1}, V_{2,y_1}) \xrightarrow{\mathrm{Res}^{-1}_{y_1}} \mathrm{Hom}(V_1, V_2(y_1)) \xrightarrow{\mathrm{ev}_{y_2}} \mathrm{Hom}(V_{1,y_2}, V_{2,y_2})$$

where the map

$$\mathrm{Res}_y : \mathrm{Hom}(V_1, V_2(y)) \widetilde{\rightarrow} \mathrm{Hom}(V_{1,y}, V_{2,y})$$

is obtained by taking the residue at a smooth point y, the map ev_y is the evaluation at a point y.
(b) Let V be a simple bundle on C. Then the tensor

$$r^{V}_{\mathcal{O}_{y_1},\mathcal{O}_{y_2}} \in \mathrm{sl}(V_{y_1}) \otimes \mathrm{sl}(V_{y_2})$$

corresponds to the composition

$$\mathrm{sl}(V_{y_1}) \xrightarrow{\mathrm{Res}^{-1}_{y_1}} H^0(C, \mathrm{ad}\, V(y_1)) \xrightarrow{\mathrm{ev}_{y_2}} \mathrm{sl}(V_{y_2})$$

where $\mathrm{ad}\, V$ is the bundle of traceless endomorphisms of V.
(c) If $V_2 \not\simeq V_1(y_2 - y_1)$ (resp. $V \not\simeq V(y_2 - y_1)$) then the tensor $r^{V_1,V_2}_{\mathcal{O}_{y_1},\mathcal{O}_{y_2}}$ in (a) (resp. $r^{V}_{\mathcal{O}_{y_1},\mathcal{O}_{y_2}}$ in (b)) is non-degenerate.

Proof. (a) Let us choose an isomorphism between the dualizing sheaf on C and $\mathcal{O}_C$. By Serre duality we have

$$\mathrm{Ext}^1(\mathcal{O}_{y_1}, V_2) \simeq \mathrm{Hom}(V_2, \mathcal{O}_{y_1})^* \simeq V_{2,y_1}.$$

Moreover, the universal extension sequence

$$0 \to V_2 \to U \to \mathrm{Ext}^1(\mathcal{O}_{y_1}, V_2) \otimes \mathcal{O}_{y_1} \to 0$$

can be identified with the canonical exact sequence

$$0 \to V_2 \to V_2(y_1) \to V_2(y_1)|_{y_1} \to 0 \tag{1.13}$$

where the isomorphism $\mathcal{O}(y_1)|_{y_1} \simeq \mathcal{O}_{y_1}$ is induced by the trivialization of the dualizing sheaf on C. Now by definition of the triple Massey products in triangulated categories we have to consider the composition map

$$\mathrm{Hom}(V_1, V_2(y_1)) \otimes \mathrm{Hom}(V_2(y_1), \mathcal{O}_{y_2}) \to \mathrm{Hom}(V_1, \mathcal{O}_{y_2})$$

and use the isomorphisms

$$\mathrm{Hom}(V_1, V_2(y_1)) \widetilde{\rightarrow} \mathrm{Hom}(V_1, V_2|_{y_1})$$
$$\mathrm{Hom}(V_2(y_1), \mathcal{O}_{y_2}) \widetilde{\rightarrow} \mathrm{Hom}(V_2, \mathcal{O}_{y_2})$$

induced by the sequence (1.13). By definition the first of these isomorphisms is given by taking the residue at y_1, so we arrive at the required description of the Massey product.
(b) The proof is analogous to (a) and is omitted.
(c) It is known (see [20]) that for any smooth point $y \in C$ the object $\mathcal{O}_y$ is spherical and the corresponding functor $T_{\mathcal{O}_y}$ is given by tensoring with the line bundle $\mathcal{O}_C(y)$. Thus, by theorem 3 the tensor $r^{V_1,V_2}_{\mathcal{O}_{y_1},\mathcal{O}_{y_2}}$ is non-degenerate if and only if

$$\mathrm{Ext}^i(V_1(y_2), V_2(y_1)) = 0$$

for $i = 0, 1$. Note that the Riemann-Roch theorem for vector bundles on C implies that

$$h^1(C, V_1^{\vee} \otimes V_2(y_1 - y_2)) = h^0(C, V_1^{\vee} \otimes V_2(y_1 - y_2)).$$

Since V_1 and $V_2(y_1 - y_2)$ are non-isomorphic simple bundles we have $\mathrm{Hom}(V_1, V_2(y_1 - y_2)) = 0$, therefore $\mathrm{Ext}^1(V_1, V_2(y_1 - y_2)) = 0$. The case of the tensor $r^V_{\mathcal{O}_{y_1},\mathcal{O}_{y_2}}$ is similar. □

Combining this theorem with theorem 2 we obtain non-degenerate solutions of the AYBE and of the CYBE associated with simple bundles on a projective curve C of arithmetic genus 1 with trivial dualizing sheaf. More precisely, we also have to choose a connected component C_0 of C in which points y_i vary. If we fix a point $y_0 \in C$ and a uniformization of $C_0 \cap C^{reg}$ compatible with the group law on the set of smooth points C^{reg} of C, then we can consider the tensor r as depending on complex parameters (two parameters in the case of the AYBE and one parameter in case of the CYBE). It is known that in the case when C is an elliptic curve one obtains all non-degenerate elliptic solutions of the CYBE by the procedure described in Theorem 4 (b). In section 3 we will construct a simple bundle of rank 2 on the union of two $\mathbb{P}^1$'s intersecting in two points. Considering points on two different components of this curve we will obtain two different trigonometric solutions of the CYBE for sl_2. In each of these cases (elliptic and trigonometric for sl_2) we also construct solutions of the AYBE specializing to the solutions of the CYBE.

2. Elliptic solutions

2.1. **Non-degenerate elliptic solutions.** Let E be an elliptic curve over a field k, V be a simple vector bundle on E, i.e. such that $\mathrm{Hom}(V, V) \simeq k$. Note that V is a 1-spherical object in the derived category of coherent sheaves on E. Assume that V has positive degree d. Then we can apply the construction of the tensor $r^{X_1X_2}_{Y_1Y_2}$ (resp. $r^X_{Y_1,Y_2}$) from section 1.2 (resp. section 1.3) to X_i varying in a family of line bundles of degree zero (resp. $X = \mathcal{O}_E$), Y_j varying in a family of bundles obtained from V be translation. Note that this is essentially equivalent to the situation of section 1.6 since applying the Fourier-Mukai transform to structure sheaves of points one gets line bundles of degree 0. Let $e \in E$ be the neutral element. We fix a trivialization of $\det V$ (top wedge power of V) at e. For every $x \in E(k)$ let us consider the following line bundle on E trivialized at e:

$$\mathcal{P}^d_x = t^*_x \det V \otimes (\det V)^{-1} \otimes (\det V)^{-1}|_x$$

where $t_x : E \to E$ is the translation by x. Note that $\mathcal{P}^d_x$ depends on V only through its degree d which is reflected in the notation. The map $x \mapsto \mathcal{P}^d_x$ is a homomorphism from $E(k)$ to the Picard group of E. Furthermore, if we denote

$$\langle x, y\rangle^d = (\mathcal{P}^d_x)|_y$$

then $\langle ?, ?\rangle^d$ is a symmetric biextension of $E \times E$. We claim that there exists a line bundle L on E such that for every $x \in E(k)$ there is a canonical isomorphism

$$t^*_{rx} V \simeq L|_x \otimes \mathcal{P}^d_x \otimes V,$$

where r is the rank of V. Indeed, since the isomorphism class of a simple vector bundle is determined by its determinant it suffices to check that $t^*_{rx}V$ and $\mathcal{P}^d_x \otimes V$ have the same determinants which is clear (in fact, using the theorem of the cube one can show that $L \simeq (\det V)^r$). Thus, for every $x, y \in E(k)$ we have a sequence of isomorphisms

$$\begin{aligned}\mathrm{Hom}(\mathcal{P}^d_x, t^*_y V) \simeq H^0(E, \mathcal{P}^d_{-x} \otimes t^*_y V) \simeq \langle x, y\rangle^d \otimes H^0(E, \mathcal{P}^d_{-x} \otimes V) \simeq \\ \langle x, y\rangle^d \otimes L^{-1}|_{-x} \otimes H^0(E, t^*_{-rx} V) \simeq \langle x, y\rangle^d \otimes L^{-1}|_{-x} \otimes H^0(E, V).\end{aligned}$$

Thus, the function

$$(x_1, x_2; y_1, y_2) \mapsto r_V(x_1, x_2; y_1, y_2) := r^{\mathcal{P}^d_{x_1},\mathcal{P}^d_{x_2}}_{t^*_{y_1}V, t^*_{y_2}V}$$

takes values in

$$\langle x_2 - x_1, y_1 - y_2\rangle^d \otimes \mathrm{End}(H^0(E, V)) \otimes \mathrm{End}(H^0(E, V))$$

while the function

$$(y_1, y_2) \mapsto r_V(y_1, y_2) := r^{\mathcal{O}}_{t^*_{y_1}V, t^*_{y_2}V}$$

takes values in $\mathrm{pgl}(H^0(E,V))\otimes\mathrm{pgl}(H^0(E,V))$. Note that $r_V(x_1,x_2;y_1,y_2)$ is defined only when $\mathcal{P}^d_{x_1}\not\simeq\mathcal{P}^d_{x_2}$ and $t^*_{y_1}V\not\simeq t^*_{y_2}V$ which happens presicely when $d(x_1-x_2)\neq 0$ and $d(y_1-y_2)\neq 0$ in E. Similarly, $r^V(y_1,y_2)$ is defined for $d(y_1-y_2)\neq 0$ in E. Also it is easy to see that $r_V(x_1,x_2;y_1,y_2)$ (resp. $r_V(y_1,y_2)$) actually depends only on the differences x_1-x_2 and y_1-y_2 (resp. on y_1-y_2). So we will use the notation

$$r_V(x;y)=r_V(0,x;0,y),$$

$$r_V(y)=r_V(0,y).$$

Now we will show that the non-degeneracy criterion of theorem 3 applies to these tensors for generic values of parameters.

Proposition 2.1. *Assume that $x,y\in E(k)$ are such that $dx\neq 0$, $dy\neq 0$, $d(dy-x)\neq 0$ (resp. $y\in E(k)$ is such that $d^2y\neq 0$).Then the tensor $r_V(x,y)$ (resp. $r_V(y)$) is non-degenerate.*

Proof. Using the action of a central extension of $\mathrm{SL}_2(\mathbb{Z})$ of $D^b(E)$ (see [14], [15]) we can find an autoequivalence $S:D^b(E)\to D^b(E)$ which sends a pair of bundles (V,t^*_yV) to the pair of sheaves $(\mathcal{O}_{y_1},\mathcal{O}_{y_2})$ for some points $y_1\neq y_2$. Then $S(\mathcal{O}_E)$ and $S(\mathcal{P}^d_x)$ are simple vector bundles of rank d. Since the twist functors $T_{\mathcal{O}_{y_i}}$ are just tensorings by $\mathcal{O}_E(y_i)$ we have only to check that

$$S(\mathcal{O}_E)(y_2)\not\simeq S(\mathcal{P}^d_x)(y_1)$$

and $S(\mathcal{O}_E)(y_2)\not\simeq S(\mathcal{O}_E)(y_1)$. Since a simple vector bundle is determined up to an isomorphism by its determinant, it suffices to check that

$$\det(S(\mathcal{O}_E))(d(y_2-y_1))\not\simeq\det(S(\mathcal{P}^d_x)).$$

$$\det(S(\mathcal{O}_E))(d(y_2-y_1))\not\simeq\det(S(\mathcal{O}_E)).$$

It is easy to see that we have an equality $y_2-y_1=\pm dy$ in the group $E(k)$. Changing S by $[-id_E]^*S$ if necessary we can assume that $y_2-y_1=dy$. Then considering the action of S on $K_0(E)$ we derive the isomorphism

$$\det(S(\mathcal{P}^d_x))\simeq\det(S(\mathcal{O}_E))(x'-e)$$

where $x'=dx$ in $E(k)$. Our assertion follows. □

Thus, in the case $k=\mathbb{C}$ using some uniformization $\pi:\mathbb{C}\to E$ we can consider the functions

$$r_V(u,v):=r_V(\pi(u),\pi(v))$$

and

$$r_V(u):=r_V(\pi(u))$$

as meromorphic solutions of the AYBE and CYBE respectively satisfying some additional conditions (namely, the unitarity and the non-degeneracy conditions).

In particular, $r_V(u)$ is a solution of CYBE satisfying all the additional conditions imposed by Belavin and Drinfeld in [2]. The explicit formulas of section 2.2 imply that $r_V(v)$ has poles at the points of the lattice $\pi^{-1}(E_d)$ (and is periodic with respect to the lattice $\pi^{-1}(0)$). In order to find the place of $r_V(u)$ in Belavin-Drinfeld classification we have to determine the automorphisms

$$A_\gamma:\mathrm{pgl}(H^0(E,V))\to\mathrm{pgl}(H^0(E,V))$$

for all $\gamma\in\pi^{-1}(E_d)$ such that

$$r_V(u+\gamma)=(A_\gamma\otimes 1)r_V(u)$$

(see Prop.4.3 of [2]). Note that by periodicity of $r_V(u)$ with respect to $\pi^{-1}(0)$ the automorphism A_γ depends only on $\pi(\gamma)\in E_d$.

Let H be the Heisenberg group associated with V. Recall that H is the central extension of E_d (the subgroup of points of order d in E) by $\mathbb{G}_m$. Points of H are pairs (x,α) where $x\in E_d$, $\alpha:V\to t^*_xV$ is an isomorphism. The space $H^0(E,V)$ is an irreducible representation of H in a natural way. This induces

a natural action of $E_d = H/\mathbb{G}_m$ on $\mathrm{pgl}(H^0(E,V))$. It is easy to see that the automorphism A_γ above is given by the action of $\pi(\gamma) \in E_d$.

The solution $r_V(u)$ gets replaced by an equivalent one if we replace V by $T(V)$ where V is any autoequivalence of $D^b(E)$ preserving $\mathcal{O}_E$. Thus, the only data on which $r_V(u)$ depends are $(d = \deg(V), r = \mathrm{rk}(V) \mod d)$. Note that the rank r is relatively prime to d since V is simple. It follows that the solutions for pgl_d are numbered by $(\mathbb{Z}/d\mathbb{Z})^*$. The choice of $r \in (\mathbb{Z}/d\mathbb{Z})^*$ precisely corresponds to a choice of a primitive d-th root of unity in Belavin-Drinfeld's picture.

2.2. **Explicit formulas.** Now we assume that $k = \mathbb{C}$ and write explicit formulas for the above solutions. The elliptic solutions of the AYBE can be expressed in terms of the Kronecker function

$$F(u,v) = \frac{\theta'_{11}(0)}{2\pi i} \cdot \frac{\theta_{11}(u+v)}{\theta_{11}(u)\theta_{11}(v)} \tag{2.1}$$

where

$$\theta_{11}(u,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n \exp(\pi i (n+\frac{1}{2})^2\tau + 2\pi i (n+\frac{1}{2})u),$$

θ'_{11} is the derivative of $\theta_{11}(u,\tau)$ with respect to u. When we want to stress the dependance of F on τ we will write $F(u,v,\tau)$. Kronecker discovered the following series expansion:

$$F(u,v) = -\sum_{(m+\frac{1}{2})(n+\frac{1}{2})>0} \mathrm{sign}(m+\frac{1}{2}) \exp(2\pi i(mn\tau + mv + nu))$$

where m,n are integers, $0 < \mathrm{Im}(u), \mathrm{Im}(v) < \mathrm{Im}(\tau)$. Let us introduce a little bit more notation. For a pair of rational numbers (p,q) we set

$$F_{p,q}(u,v) = \exp(2\pi i(pq\tau + pv + qu))F(u+p\tau, v+q\tau). \tag{2.2}$$

For $0 < \mathrm{Im}(u), \mathrm{Im}(v) < \epsilon$ where ϵ is sufficiently small, one has

$$F_{p,q}(u,v) = -\sum_{(m,n)\in\mathbb{Z}^2+(p,q),(m+\epsilon)(n+\epsilon)>0} \mathrm{sign}(m+\epsilon) \exp(2\pi i(mn\tau + mv + nu)).$$

Note that we have the symmetry relation

$$F_{p,q}(u,v) = F_{q,p}(v,u).$$

This kind of series appear in the computation of triple Fukaya compositions corresponding to the Massey products defining $r_V(u,v)$.

Let us consider first the case $r = 1$, so $V = L$ is a line bundle of degree d. We denote by $(e_i, i \in \mathbb{Z}/d\mathbb{Z})$ the natural basis in $H^0(E,L)$ consisting of theta-functions with characteristics. Let e_i^* be the dual basis in $H^0(E,L)^*$. Then using the correspondence between our Massey products and triple Fukaya compositions (see [16]) one can derive the following formula:

$$m_3(e_i, e_j^*, e_k) = F_{\frac{i-j}{d},\frac{j-k}{d}}(du, -dv, d\tau)e_{i-j+k}.$$

Hence,

$$r_L(u,v) = \sum_{j-i=i'-j'} F_{\frac{j-i}{d},\frac{i-j'}{d}}(du,-dv,d\tau)e_{ij}\otimes e_{i'j'} \tag{2.3}$$

where e_{ij} is the standard basis in the matrix algebra $\mathrm{Mat}(d,\mathbb{C})$. In the simplest case when $d = 1$ we obtain just the function $F(u,-v)$, so the AYBE in this case specializes to the following identity:

$$F(-u',v)F(u+u',v+v') - F(u+u',v')F(u,v) + F(u,v+v')F(u',v') = 0. \tag{2.4}$$

To find formulas for the corresponding solutions of the CYBE we project the tensor $r_L(u,v) \in \mathrm{Mat}(d,\mathbb{C}) \otimes \mathrm{Mat}(d,\mathbb{C})$ to $\mathrm{sl}_d \otimes \mathrm{sl}_d$ and then set $u = 0$. Using the above formula for $r_L(u,v)$ we obtain

$$\overline{r}(v) := (\mathrm{pr} \otimes \mathrm{pr})(r_L(u,v)) = \sum_{j-i=i'-j'\neq 0} F_{\frac{j-i}{d},\frac{i-j'}{d}}(du,-dv,d\tau)e_{ij} \otimes e_{i'j'} + \sum_{i,i'} G_{i-i'}(du,-dv,d\tau)e_{ii} \otimes e_{i'i'} \tag{2.5}$$

where

$$G_j(x,y,\tau) = F_{0,\frac{j}{d}}(x,y,\tau) - \frac{1}{d} \cdot \sum_{k\in\mathbb{Z}/d\mathbb{Z}} F_{0,\frac{k}{d}}(x,y,\tau)$$

When passing to the limit $u \to 0$ in the formula (2.5) we are going to use the following relation between the Kronecker function $F(u,v)$ and the Weierstrass zeta-function observed in [19]. Let $\zeta(x) = \zeta(x,\tau)$ denotes the Weierstrass zeta-function associated with the lattice $\mathbb{Z} + \mathbb{Z}\tau$. Then according to [19], Cor.1.2, we have

$$\left(2\pi i F(x,y) - \frac{1}{x}\right)|_{x=0} = \zeta(y) - y\eta_1$$

where $\eta_1 = 2\zeta(\frac{1}{2})$. It follows that for any function $g : \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}$ with $\sum_{j\in\mathbb{Z}/d\mathbb{Z}} g(j) = 0$ we have

$$\left(2\pi i \sum_{j\in\mathbb{Z}/d\mathbb{Z}} g(j)F_{0,\frac{j}{d}}(x,y)\right)|_{x=0} = \sum_{j\in\mathbb{Z}/d\mathbb{Z}} g(j)\zeta(y + \frac{j}{d}\tau) + (2\pi i - \eta_1\tau) \sum_{j\in\mathbb{Z}/d\mathbb{Z}} \frac{g(j)j}{d}.$$

Using the Legendre relation $\eta_1\tau - \eta_2 = 2\pi i$, where $\eta_2 = 2\zeta(\frac{\tau}{2})$, we can rewrite this formula as follows:

$$\left(2\pi i \sum_{j\in\mathbb{Z}/d\mathbb{Z}} g(j)F_{0,\frac{j}{d}}(x,y)\right)|_{x=0} = \sum_{j\in\mathbb{Z}/d\mathbb{Z}} g(j)\zeta_{0,\frac{j}{d}}(y,\tau). \tag{2.6}$$

where we use the notation (6.1). In particular, we obtain

$$2\pi i G_j(0,y,\tau) = \zeta_{0,\frac{j}{d}}(y,\tau) - \frac{1}{d} \cdot \sum_{k\in\mathbb{Z}/d\mathbb{Z}} \zeta_{0,\frac{k}{d}}(y,\tau).$$

Now the expression for the solutions of the CYBE takes form

$$\begin{aligned} 2\pi i\overline{r}(v) = \textstyle\sum_{j-i=i'-j'\neq 0} 2\pi i F_{\frac{j-i}{d},\frac{i-j'}{d}}(0,-dv,d\tau)e_{ij} \otimes e_{i'j'} + \\ \textstyle\sum_{i,i'}(\zeta_{0,\frac{i-i'}{d}}(-dv,d\tau) - \frac{1}{d}\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \zeta_{0,\frac{k}{d}}(-dv,d\tau))e_{ii} \otimes e_{i'i'}. \end{aligned} \tag{2.7}$$

Using formulas (6.2) and (6.4) we can rewrite this as follows:

$$\begin{aligned} 2\pi i\overline{r}(v) = \textstyle\sum_{j-i=i'-j'\neq 0} \sum_{a\in\mathbb{Z}/d\mathbb{Z}} \exp(-2\pi i\frac{a(j-i)}{d})[\zeta_{\frac{a}{d},\frac{i-j'}{d}}(-v,\tau) - \zeta_{\frac{a}{d},0}(\frac{i-j}{d}\tau,\tau)]e_{ij} \otimes e_{i'j'} + \\ \textstyle\sum_{i,i'}[\frac{1}{d}\sum_{a\in\mathbb{Z}/d\mathbb{Z}} \zeta_{\frac{a}{d},\frac{i-i'}{d}}(-v,\tau) - \frac{1}{d^2}\sum_{a,b\in\mathbb{Z}/d\mathbb{Z}} \zeta_{\frac{a}{d},\frac{b}{d}}(-v,\tau)]e_{ii} \otimes e_{i'i'}. \end{aligned} \tag{2.8}$$

The case $r > 1$ can be easily reduced to the case $r = 1$ using a representation of the bundle V as the direct image of a line bundle L under the isogeny $\mathbb{C}/\mathbb{Z} + r\tau\mathbb{Z} \to \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. It is easy to see that in this situation one has

$$r_V(u,v;\tau) = r_L(ru,v,r\tau).$$

3. Trigonometric solutions for sl_2

It turns out that computations of Massey products are easier in the case of a reducible curve. Also in order to obtain all non-degenerate solutions of the CYBE for sl_2 it is necessary to consider a curve with 2 components. Because of this we chose to study the solutions of the AYBE and the CYBE arising from simple bundles of rank 2 on such a curve.

3.1. **Construction of simple bundles of rank 2 on a reducible curve.** Let $C = C_1 \cup C_2$ be the union of two $\mathbb{P}^1$'s glued (transversally) by two points. In other words, $C_1 = C_2 = \mathbb{P}^1$ and the point 0 (resp. ∞) on C_1 is identified with the point 0 (resp. ∞) on C_2. A vector bundle V on C is given by the following data:

$$(V_1, V_2, \alpha_0 : V_{1,0} \widetilde{\rightarrow} V_{2,0}, \alpha_\infty : V_{1,\infty} \widetilde{\rightarrow} V_{2,\infty})$$

where V_i is a bundle on C_i, $i = 1, 2$, $V_{i,x}$ denotes the fiber of V_i at the point x. For each $\lambda \in k^*$ let us define the rank-2 bundle V^λ on C as follows:

$$V_1^\lambda = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1},$$

$$V_2^\lambda = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1),$$

$$\alpha_0 = \mathrm{id},\ \alpha_\infty = S_\lambda := \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}.$$

Here we use the trivialization of $\mathcal{O}_{\mathbb{P}^1}(1)$ at 0 (resp. ∞) induced by the standard trivialization of $\mathcal{O}_{\mathbb{P}^1}(1)$ on the complement to ∞ (resp. 0).

Lemma 3.1. *The bundle V^λ is simple.*

Proof. An endomorphism of V^λ is given by a pair of endomorphisms $f_1 : V_1^\lambda \to V_1^\lambda$ and $f_2 : V_2^\lambda \to V_2^\lambda$, such that $f_1(0) = f_2(0)$ (this follows from $\alpha_0 = \mathrm{id}$) and

$$f_2(\infty) S_\lambda = S_\lambda f_1(\infty).$$

Note that f_1 has constant coefficients so $f_1 = f_1(0) = f_1(\infty)$. The endomorphism f_2 is lower-triangular (since $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}) = 0$), hence, f_1 and $S_\lambda f_1 S_\lambda^{-1}$ are both lower-triangular which implies that f_1 is diagonal. Notice that the diagonal part of f_2 is constant, so we deduce that

$$f_1 = S_\lambda f_1 S_\lambda^{-1}$$

which is possible only if f_1 is proportional to the identity. Finally, it is easy to see that f_2 is completely determined by $f_2(0)$ and $f_2(\infty)$, so the only endomorphisms of V are scalar multiples of the identity. □

3.2. **Computation.** Now we are going to apply theorem 4 to compute the solutions of the AYBE and the CYBE associated with bundles V^λ. For this we have to describe the space of morphisms $\mathrm{Hom}(V^{\lambda_1}, V^{\lambda_2}(y))$, where $\lambda_i \in k^*$, y is a smooth point of C. There are two different cases to consider depending on whether $y \in C_1$ or $y \in C_2$.
Case 1. $y \in C_1$. Then a morphism $V^{\lambda_1} \to V^{\lambda_2}(y)$ is given by a pair of morphisms on $\mathbb{P}^1$:

$$A : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(y) \oplus \mathcal{O}_{\mathbb{P}^1}(y),$$

$$B : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

satisfying the conditions $A_0 = B_0$ and

$$S_{\lambda_2} A_\infty = B_\infty S_{\lambda_1}.$$

We claim that such a morphism is completely determined by B which can be arbitrary. Indeed, considering A as an endomorphism of $\mathcal{O}_{\mathbb{P}^1}^2$ with a pole of the first order at y we can write it uniquely in the form

$$A = \frac{1}{z-y} \cdot A' + \frac{z}{z-y} \cdot A''$$

where A', A'' are some regular endomorphisms of $\mathcal{O}_{\mathbb{P}^1}^2$, $z = \frac{z_1}{z_0}$. Now we have

$$A_0 = -\frac{A'}{y},\ A_\infty = A'',$$

hence A is uniquely recovered from A_0 and A_∞. Thus, to every $B \in \mathrm{End}(\mathcal{O}_{\mathcal{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ we can associate the morphism $(A, B) : V^{\lambda^1} \to V^{\lambda_2}(y)$ with

$$A = \frac{y}{y-z} \cdot B_0 + \frac{z}{z-y} \cdot S_{\lambda_2}^{-1} B_\infty S_{\lambda_1}.$$

In this description the residue morphism

$$\mathrm{Res}_y : \mathrm{Hom}(V^{\lambda_1}, V^{\lambda_2}(y)) \to \mathrm{Mat}(2, k)$$

is given by the formula

$$B \mapsto S_{\lambda_2}^{-1} B_\infty S_{\lambda_1} - B_0$$

(here we use a local trivialization of ω_C given by the form $\frac{dz}{z}$). Let us write

$$B = \begin{pmatrix} a & 0 \\ bz_0 + cz_1 & d \end{pmatrix}.$$

Then we have

$$\mathrm{Res}_y : B \mapsto \begin{pmatrix} d-a & \lambda_1 c \\ -b & \lambda_1\lambda_2^{-1} a - d \end{pmatrix}.$$

On the other hand, if $y_1, y_2 \in C_1$ are distinct points then after applying the above computation to $y = y_1$ we can consider the evaluation map

$$\mathrm{ev}_{y_2} : \mathrm{Hom}(V^{\lambda_1}, V^{\lambda_2}(y_1)) \to \mathrm{Mat}(2, k) : B \mapsto \frac{y_1}{y_1 - y_2} \cdot B_0 + \frac{y_2}{y_2 - y_1} \cdot S_{\lambda_2}^{-1} B_\infty S_{\lambda_1}.$$

Thus, we can compute the map

$$\mathrm{ev}_{y_2} \circ \mathrm{Res}_{y_1}^{-1} : \mathrm{Mat}(2, k) \to \mathrm{Mat}(2, k) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$$

$$\frac{y_1}{y_1 - y_2} \cdot \begin{pmatrix} \frac{a+d}{\lambda_1\lambda_2^{-1} - 1} & 0 \\ -c & \frac{\lambda_1\lambda_2^{-1}a + d}{\lambda_1\lambda_2^{-1} - 1} \end{pmatrix} + \frac{y_2}{y_2 - y_1} \cdot \begin{pmatrix} \frac{\lambda_1\lambda_2^{-1}a + d}{\lambda_1\lambda_2^{-1} - 1} & b \\ 0 & \lambda_1\lambda_2^{-1} \cdot \frac{a+d}{\lambda_1\lambda_2^{-1} - 1} \end{pmatrix}.$$

Note that this map depends only on $\lambda = \lambda_1\lambda_2^{-1}$ and $\mu = y_1 y_2^{-1}$. Thus, from theorem 4 we obtain the following solution of the AYBE (where λ and μ should be considered as multiplicative variables which are exponents of the additive variables appearing in (0.1)):

$$\begin{aligned} r(\lambda, \mu) = \tfrac{1}{(1-\lambda)(1-\mu)} \left((\mu e_{11} - e_{22}) \otimes (e_{11} + \lambda e_{22}) + (-\lambda e_{11} + \mu e_{22}) \otimes (e_{11} + e_{22}) \right) + \\ \tfrac{1}{1-\mu} e_{21} \otimes e_{12} + \tfrac{\mu}{1-\mu} e_{12} \otimes e_{21}. \end{aligned} \tag{3.1}$$

Projecting this tensor to sl_2 we obtain the corresponding solution of the CYBE:

$$r(\mu) = \frac{1+\mu}{4(1-\mu)} h \otimes h + \frac{e_{21} \otimes e_{12} + \mu e_{12} \otimes e_{21}}{1-\mu}. \tag{3.2}$$

where $h = e_{11} - e_{22}$.

Case 2. $y \in C_2$. Then a morphism $V^{\lambda_1} \to V^{\lambda_2}(y)$ is given by a pair of morphisms on $\mathbb{P}^1$:

$$A : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1},$$

$$B : \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))(y)$$

satisfying the conditions $A = B_0$ and $S_{\lambda_2} A = B_\infty S_{\lambda_1}$. Such a morphism is completely determined by B which should satisfy the condition

$$B_0 = S_{\lambda_2}^{-1} B_\infty S_{\lambda_1}.$$

Considering B as an endomorphism of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ with a pole of the first order at y we can write it in the form

$$B = \frac{1}{z-y} \cdot B' + \frac{z}{z-y} \cdot B'' + \begin{pmatrix} 0 & \frac{t}{z_1 - z_0 y} \\ 0 & 0 \end{pmatrix}$$

where B' and B'' are regular endomorphisms of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $t \in k$, $\frac{1}{z_1 - z_0 y}$ is a section of $\mathcal{O}_{\mathbb{P}^1}(-1)$ with the pole at y. However, this presentation is non-unique: we can add to B' a lower-triangular endomorphism vanishing at 0 and change B'' appropriately. To get rid of this ambiguity we impose the condition that B'_∞ is diagonal. Then B' and B'' are unique. Furthermore, in this case B' is uniquely determined by B'_0. On the other hand, we have

$$B_0 = -\frac{B'_0 + te_{12}}{y},$$

$$B_\infty = B''_\infty + te_{12},$$

hence we get the equation

$$B'_0 + te_{12} = -yS_{\lambda_2}^{-1}(B''_\infty + te_{12})S_{\lambda_1}.$$

Solving this equation for B'_0 and t we obtain that for

$$B'' = \begin{pmatrix} a'' & 0 \\ b''z_0 + c''z_1 & d'' \end{pmatrix}$$

one has $t = -y\lambda_1 c''$ and

$$B'_0 = -y \cdot \begin{pmatrix} d'' & 0 \\ -y\lambda_1\lambda_2^{-1}c'' & \lambda_1\lambda_2^{-1}a'' \end{pmatrix}.$$

Thus, all the data can be recovered from B'' which can be arbitrary. Now we can compute the map Res_y. Notice that the difference from the previous case is that we have to choose a trivialization of V^{λ_1} and V^{λ_2} at y (since now y belongs to the component C_2 on which these bundles are non-trivial). Our choice for V^λ will correspond to the trivialization of $\mathcal{O}_{\mathbb{P}^1}(1)$ at y given by the non-vanishing section $f_\lambda^{-1}z_0$, where f_λ is some invertible function on $\mathbb{P}^1 - \{0, \infty\}$. Then using B'' as a coordinate on $\mathrm{Hom}(V^{\lambda_1}, V^{\lambda_2})$ we obtain

$$\mathrm{Res}_y(B'') = \frac{B'_y}{y} + B''_y + \frac{t}{y}e_{12} = \begin{pmatrix} a'' - d'' & -f_{\lambda_1}^{-1}(y)\lambda_1 c'' \\ f_{\lambda_2}(y)(b'' + y(1+\lambda_1\lambda_2^{-1})c'') & d'' - \lambda_1\lambda_2^{-1}a'' \end{pmatrix}$$

On the other hand, using the above construction for $y = y_1$ and taking a point $y_2 \neq y_1$ in C_2 we can compute the evaluation map

$$\begin{aligned} &\mathrm{ev}_{y_2} : \mathrm{Hom}(V^{\lambda_1}, V^{\lambda_2}(y_1)) \to \mathrm{Mat}(2,k) : \\ &B'' \mapsto \frac{1}{y_2 - y_1}B'_{y_2} + \frac{y_2}{y_2 - y_1}B''_{y_2} + \frac{t}{y_2 - y_1}e_{12} = \\ &\frac{y_2}{y_2 - y_1} \cdot \begin{pmatrix} a'' - y_1y_2^{-1}d'' & -f_{\lambda_1}^{-1}(y_2)\lambda_1 c'' \\ f_{\lambda_2}(y_2)(b'' + (y_2 + y_1^2y_2^{-1}\lambda_1\lambda_2^{-1})c'') & d'' - y_1y_2^{-1}\lambda_1\lambda_2^{-1}a'' \end{pmatrix}. \end{aligned}$$

Finally, we compute the map

$$\begin{aligned} &\mathrm{ev}_{y_2} \circ \mathrm{Res}_{y_1}^{-1} : \mathrm{Mat}(2,k) \to \mathrm{Mat}(2,k) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \\ &\frac{1}{1-\mu} \cdot \begin{pmatrix} \frac{(1-\mu\lambda)a + (1-\mu)d}{1-\lambda} & f_{\lambda_1}(y_2)^{-1}f_{\lambda_1}(y_1)b \\ f_{\lambda_2}(y_2)[f_{\lambda_1}(y_1)(y_1 - y_2)(1-\mu\lambda)\lambda_1^{-1}b + f_{\lambda_2}(y_1)^{-1}c] & \frac{(1-\mu)\lambda a + (1-\mu\lambda)d}{1-\lambda} \end{pmatrix} \end{aligned}$$

where we denoted $\lambda = \lambda_1\lambda_2^{-1}$, $\mu = y_1y_2^{-1}$. Now we observe that if we set

$$f_\lambda(y) = \lambda^{\frac{1}{2}}y^{-\frac{1}{2}}$$

then the above matrix will depend only on λ and μ. Thus, we obtain the following solution of the AYBE (in the multiplicative notation):

$$\begin{aligned} r(\lambda, \mu) = &\tfrac{1-\lambda\mu}{(1-\lambda)(1-\mu)}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + \tfrac{\lambda e_{11}\otimes e_{22} + e_{22}\otimes e_{11}}{1-\lambda} + \\ &\tfrac{\mu^{-\frac{1}{2}}}{1-\mu}e_{21} \otimes e_{12} + \tfrac{\mu^{\frac{1}{2}}}{1-\mu}e_{12} \otimes e_{21} + ((\lambda\mu)^{\frac{1}{2}} - (\lambda\mu)^{-\frac{1}{2}})e_{21} \otimes e_{21}. \end{aligned} \tag{3.3}$$

Applying the projection to sl_2 and setting $\lambda = 1$ we get the following solution of the CYBE:

$$r(\mu) = \frac{1+\mu}{4(1-\mu)} h \otimes h + \frac{\mu^{-\frac{1}{2}} e_{21} \otimes e_{12} + \mu^{\frac{1}{2}} e_{12} \otimes e_{21}}{1-\mu} + (\mu^{\frac{1}{2}} - \mu^{-\frac{1}{2}}) e_{21} \otimes e_{21}. \tag{3.4}$$

where $h = e_{11} - e_{22}$. It is easy to see that our solutions (3.2) and (3.4) are equivalent to the solutions (6.9) and (6.10) in [2] which represent two distinct equivalence classes of non-degenerate trigonometric solutions of the CYBE for sl_2. Note that we actually constructed a solution r_{y_1,y_2} of the equation (1.9) depending on parameters $y_1, y_2 \in C^{reg}$ (with a pole at $y_1 = y_2$) which specializes to the above two solutions when y_i vary in one of the two components of C.

4. Scalar solutions of AYBE

In this section we are going to study the equation (0.1) in the case when $n = 1$, i.e. when $r(u, v)$ is $\mathbb{C}$-valued.

Theorem 5. *Let $r(u, v)$ be a non-zero meromorphic function in the neighborhood of $(0, 0)$ satisfying the equations*

$$r(-u', v)r(u + u', v + v') - r(u + u', v')r(u, v) + r(u, v + v')r(u', v') = 0,$$
$$r(-u, -v) = -r(u, v).$$

Then there exist constants $c_1, c_3, c_4 \in \mathbb{C}^*$ and $c_2 \in \mathbb{C}$ such that $c_1 \exp(c_2 uv) r(c_3 u, c_4 v)$ is one of the following functions:
1) $F_\tau(u, v) = F(u, v, \tau)$ (Kronecker's function),

2) $F_\infty(u, v) := \frac{\exp(v)-\exp(u)}{(\exp(u)-1)(\exp(v)-1)}$,

3) $\frac{a}{u} + \frac{b}{v}$, $a, b \in \mathbb{C}$.

Proof. Assume first the divisor of poles of r doesn't contain $u = 0$. Substituting $u = 0$ in the equation we obtain

$$r(-u', v)r(u', v + v') - r(u', v')r(0, v) + r(0, v + v')r(u', v') = 0. \tag{4.1}$$

Substiting $u' = 0$ we get

$$(r(0, v) + r(0, v'))r(0, v + v') = r(0, v)r(0, v').$$

Note that $r(0, v)$ is not identically zero: otherwise (4.1) would imply that $r(u, v)$ is identically zero. Hence, we can write the last equation as

$$\frac{1}{r(0, v + v')} = \frac{1}{r(0, v)} + \frac{1}{r(0, v')}.$$

Thus, multiplying r by a constant we can assume that

$$r(0, v) = \frac{1}{v}.$$

Substituting this in (4.1) we obtain

$$r(-u', v)r(u', v + v') = r(u', v')(\frac{1}{v} - \frac{1}{v + v'}).$$

Using the equality $r(-u', v) = -r(u', -v)$ we can rewrite this as follows

$$-r(u', -v)r(u', v + v')v(v + v') = r(u', v')v'.$$

This implies that

$$r(u, v)v = \exp(c(u)v) \tag{4.2}$$

for some meromorphic function $c(u)$. Substituting this in the original equation we get

$$\frac{\exp(c(-u')v + c(u+u')(v+v'))}{v(v+v')} - \frac{\exp(c(u+u')v' + c(u)v)}{vv'} + \frac{\exp(c(u)(v+v') + c(u')v')}{v'(v+v')} = 0$$

Multiplying by $v+v'$ and collecting terms with $1/v$ and $1/v'$ we get

$$\frac{\exp((c(u+u') - c(u) - c(u'))v) - 1}{v} = \frac{1 - \exp((c(u) + c(u') - c(u+u'))v')}{v'}.$$

This immediately implies that

$$c(u+u') = c(u) + c(u'),$$

hence

$$r(u,v) = \frac{\exp(cuv)}{v}$$

for some constant c which leads to case 3).

Now let us assume that r has pole along $u = 0$ of order $k > 0$. Writing r in the form $r(u,v) = \sum_{i \le -k} r_i(v)u^i$ and substituting in the equation we obtain that

$$r_{-k}(v)r_{-k}(v+v')(-u')^{-k}(u+u')^{-k} - r_{-k}(v')r_{-k}(v)u^{-k}(u+u')^{-k} + r_{-k}(v+v')r_{-k}(v')u^{-k}(u')^{-k} = 0.$$

It is easy to see that this is possible only if $k = 1$ and $r_{-1}(v)$ is constant. Multiplying r be a constant we can assume that

$$r(u,v) = \frac{1}{u} + r_0(v) + r_1(v)u + r_2(v)u^2 + \dots$$

Note that similar arguments work for v instead of u, so we can assume that $r_0(v)$ has pole of order 1 are zero.

Now we claim that the terms r_i with $i \ge 2$ are uniquely determined by r_0 and r_1. Indeed, let us check that the term r_n for $n \ge 2$ can be recovered from the previous term. Collecting terms of the main equation which have total degree $n-1$ in u and u' we get

$$r_n(v)[\frac{(-u')^n}{u+u'} - \frac{u^n}{u+u'}] + r_n(v')[-\frac{(u+u')^n}{u} + \frac{(u')^n}{u}] + r_n(v+v')[-\frac{(u+u')^n}{u'} + \frac{u^n}{u'}] = \dots$$

where the RHS contains only r_i with $i \le n-1$. It is easy to check that if $n \ge 3$ then the polynomials in u, u'

$$\frac{(-u')^n - u^n}{u+u'};\ \frac{-(u+u')^n + (u')^n}{u};\ \frac{-(u+u')^n + u^n}{u'}$$

are linearly independent (e.g. one can check this by looking at coefficients with u^{n-1}, $u^{n-2}u'$ and $(u')^{n-1}$). Therefore, for $n \ge 3$ the term r_n is recovered from the previous terms. For $n = 2$ the above equation takes form

$$-u(r_2(v) + r_2(v')) + u'(r_2(v) - r_2(v+v')) = \dots,$$

hence, r_2 is uniquely recovered from r_0 and r_1. For $n = 1$ we get the following relation

$$r(-u',v)r(u+u',v+v') - r(u+u',v')r(u,v) + r(u,v+v')r(u',v') = 0,$$

$$r_0(v)r_0(v+v') - r_0(v')r_0(v) + r_0(v+v')r_0(v') = r_1(v) + r_1(v') + r_1(v+v') \tag{4.3}$$

Using the rescaling of the form

$$r(u,v) \mapsto c \cdot \exp(c'uv)r(cu, c''v)$$

we can achieve rescaling of r_0 of the form

$$r_0(v) \mapsto cr_0(c''v) + c'v.$$

Thus, we can assume that the Laurent expansion of r_0 at 0 has form

$$r_0(v) = \frac{1}{v} + c_3v^3 + c_5v^5 + \dots \tag{4.4}$$

where c_3 is equal to 1 or 0 (recall that r_0 is odd). Note that the LHS in (4.3) doesn't have pole at $v = 0$. Hence, r_1 is regular at 0 and taking the limit of (4.3) as $v \to 0$ we get

$$r_0'(v') + r_0(v')^2 = r_1(0) + 2r_1(v').$$

Using the Laurent expansion of r_0 at 0 we see that the LHS of this equality tends to zero as $v' \to 0$. Hence, $r_1(0) = 0$ and we get

$$r_1(v') = \frac{1}{2}(r_0'(v') + r_0(v')^2).$$

In particular, r_1 is determined by r_0. Substituting this expression for r_1 into (4.3) we obtain the following functional equation on r_0:

$$2r_0(v)r_0(v+v') - 2r_0(v')r_0(v) + 2r_0(v+v')r_0(v') = r_0'(v) + r_0'(v') + r_0'(v+v') + r_0(v)^2 + r_0(v')^2 + r_0(v+v')^2,$$

which can be rewritten as

$$(r_0(v) + r_0(v') - r_0(v+v'))^2 + r_0'(v) + r_0'(v') + r_0'(v+v') = 0. \tag{4.5}$$

We are looking for solutions of this equation which are meromorphic in the neighborhood of zero and have form (4.4).[4] Substituting the expansion (4.4) in the equation one can easily see that any solution is uniquely determined by the coefficients (c_3, c_5). The rescaling $r_0(v) \mapsto cr_0(cv)$ for $c \in \mathbb{C}^*$ leads to the rescaling $(c_3, c_5) \mapsto (c^4 c_3, c^6 c_5)$. Note that there is a (unique) solution with $c_3 = c_5 = 0$, namely, $r_0(v) = \frac{1}{v}$ (this corresponds to $r(u, v) = \frac{1}{u} + \frac{1}{v}$), so from now on we will assume that $(c_3, c_5) \neq (0, 0)$. Then up to rescaling a solution r_0 is characterized by the parameter

$$C(r_0) = \frac{c_5^2}{c_3^3}$$

which takes values in $\mathbb{C} \cup \infty$. Now we claim that from solutions $r(u, v) = 2\pi i F_\tau(u, v)$ one gets all values of $C(r_0)$ except for $-\frac{20}{49}$ while from the trigonometric solution $r(u, v) = F_\infty(u, v)$ one gets the exceptional value $-\frac{20}{49}$. Indeed, the Laurent expansion $2\pi i F_\tau(u, v)$ has form

$$2\pi i F_\tau(u, v) = \frac{1}{u} + [\frac{1}{v} - 2G_2(\tau)v - G_4(\tau)\frac{v^3}{3} - G_6(\tau)\frac{v^5}{60} + \dots] + \dots$$

where

$$G_k = -\frac{B_k}{2k} + \sum_{m,n \geq 1} m^{k-1} q^{mn}$$

are the Eisenstein series (here $q = \exp(2\pi i \tau)$). Thus, for this solution we have

$$C(r_0) = -\frac{27 G_6(\tau)^2}{60^2 G_4(\tau)^3}.$$

Recall that the j-invariant is defined by the formula

$$j(\tau) = \frac{g_2(\tau)^3}{g^2(\tau)^3 - 27 g_3(\tau)^2}$$

where

$$g_2(\tau) = 60 \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^4},$$

$$g_3(\tau) = 140 \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^6}.$$

We have the following relations:

$$G_4(\tau) = \frac{g_2(\tau)}{20(2\pi)^4},$$

[4] Meromorphic solutions of (4.5) were described by L. Carlitz in [4]. For completeness we give an independent argument.

$$G_6(\tau) = -\frac{3g_3(\tau)}{7(2\pi)^6}.$$

It follows that

$$C(r_0) = -\frac{20}{49}(1 - j(\tau)^{-1}).$$

Since $j(\tau)$ takes all complex values (including 0), we obtain all values of the parameter $C(r_0)$ (including ∞) except for $-\frac{20}{49}$. Finally, for the solution $r(u,v) = F_\infty(u,v)$ we obtain

$$r_0(v) = \frac{\coth(v/2)}{2} = \sum_{n\geq 0} \frac{B_n}{n!} v^{n-1},$$

hence

$$C(r_0) = (\frac{B_6}{6!})^2 \cdot (\frac{4!}{B_4})^3 = -\frac{20}{49}.$$

□

5. Reconstructing solutions of AYBE from solutions of CYBE

Recall that according to Lemma 1.2 if $r(u,v)$ is a unitary solution of the AYBE then the limit $\overline{r}(v) := (\mathrm{pr}\otimes\mathrm{pr})(r(u,v))|_{u=0}$ (if exists) is a solution of the CYBE with values in sl_n. In this section we study the question to which extent $r(u,v)$ is determined by $\overline{r}(v)$.

Theorem 6. *Consider unitary solutions of the AYBE with values in* $\mathrm{Mat}_n(\mathbb{C})$ *which have Laurent expansion near* $u = 0$ *of the form*

$$r(u,v) = \frac{1\otimes 1}{u} + r_0(v) + r_1(v)u + \dots .$$

Assume that the corresponding solution $\overline{r}(v) := (\mathrm{pr}\otimes\mathrm{pr})(r_0(v))$ *of the CYBE has no infinitesimal symmetries and that the tensor* $\overline{r}(v)$ *has rank* > 2 *for generic* v*. Then* $r(u,v)$ *can be uniquely recovered from* $\overline{r}(v)$ *up to a rescaling* $r(u,v) \mapsto \exp(cuv)r(u,v)$*, where* $c \in \mathbb{C}$*. In other words, two unitary solutions of the AYBE in the above form differ by a factor of the form* $\exp(cuv)$ *if and only if the corresponding solutions of the CYBE are equal.*

Proof. First the same argument as in the proof of Theorem 5 shows that $r(u,v)$ is uniquely determined by terms r_0, r_1 and r_2. Furthermore, we have an equation

$$[r_2^{12}(v) - 2r_2^{23}(v') - r_2^{13}(v+v')]\cdot u' - [r_2^{12}(v) + r_2^{23}(v') + 2r_2^{13}(v+v')]\cdot u = \dots$$

where the RHS depends only on r_0 and r_1. Hence, each term in the LHS can be recovered from r_0 and r_1. Therefore, the same is true for the expression $r_2^{12}(v) - r_2^{23}(v')$, hence for $r_2(v)$. The terms r_0 and r_1 are related by the equation

$$r_0^{12}(v)r_0^{13}(v+v') - r_0^{23}(v')r_0^{12}(v) + r_0^{13}(v+v')r_0^{23}(v') = r_1^{12}(v) + r_1^{23}(v') + r_1^{13}(v+v'). \tag{5.1}$$

We claim that r_1 is uniquely determined by r_0. Indeed, let $v \mapsto s(v)$ be a $\mathrm{Mat}_n(\mathbb{C})\otimes\mathrm{Mat}_n(\mathbb{C})$-valued meromorphic function in a neighborhood of zero such that $s^{21}(-v) = s(v)$ and

$$s^{12}(v) + s^{23}(v') + s^{13}(v+v') = 0.$$

We have to prove that s is zero. Applying $\mathrm{pr}\otimes\mathrm{id}\otimes\mathrm{id}$ to the equation we immediately deduce that

$$(\mathrm{pr}\otimes\mathrm{id})(s(v)) = 0.$$

Similarly, $(\mathrm{id}\otimes\mathrm{pr})(s(v)) = 0$, hence $s(v) = f(v)\cdot 1\otimes 1$ where $f(v)$ is an even meromorphic function satisfying $f(v) + f(v') + f(v+v') = 0$. Hence, $f = 0$.

It remains to show that $r_0(v)$ is uniquely determined by $\overline{r}(v) = (\mathrm{pr}\otimes\mathrm{pr})(r_0(v))$ up to a summand of the form $cv\cdot 1\otimes 1$, where $c\in\mathbb{C}$, provided that $\overline{r}(v)$ has no infinitesimal symmetries. Let $(\widetilde{r}_0(v), \widetilde{r}_1(v))$

be another solution of the equation (5.1) such that $\widetilde{r}_0^{21}(-v) = -\widetilde{r}(v)$, $\widetilde{r}_1^{21}(-v) = \widetilde{r}_1(v)$. We claim that if $(\mathrm{pr}\otimes\mathrm{pr})(\widetilde{r}_0(v)) = \overline{r}(v)$ then $\widetilde{r}_0(v) = r_0(v)$. Indeed, we can write

$$\widetilde{r}_0(v) = r_0(v) + \phi^1(v) - \phi^2(-v) + \psi(v)\cdot 1\otimes 1$$

for some unique $\mathrm{sl}_n(\mathbb{C})$-valued function $\phi(v)$ and some scalar function $\psi(v)$. Let us denote the LHS of the equation (5.1) by $LHS(r)$. Then we have

$$0 = (\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})(LHS(\widetilde{r}) - LHS(r)) =$$
$$\overline{r}^{12}(v)\cdot(\phi^3(-v') - \phi^3(-v-v')) + \overline{r}^{23}(v')\cdot(\phi^1(v+v') - \phi^1(v)) + \overline{r}^{13}(v+v')\cdot(\phi^2(v') - \phi^2(-v)).$$

If the function $\phi(v)$ is not constant then contracting this equation with a generic functional in the third component we derive that $\overline{r}(v)$ is a sum of two decomposable tensors which contradicts our assumption. Hence, the function $\phi(v)$ has a constant value $\phi\in\mathrm{sl}_n(\mathbb{C})$. Now applying the projection $\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{id}$ to the difference of equations (5.1) for $\widetilde{r}$ and r we get the equation

$$(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{id})(\widetilde{r}_1^{12}(v) - r_1^{12}(v)) = (\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{id})(r_0^{12}(v)\phi^1 - \phi^2 r_0^{12}(v)) - \phi^1\phi^2 + (\psi(v+v') - \psi(v'))\cdot\overline{r}^{12}(v) \tag{5.2}$$

This is possible only if $\psi(v+v') - \psi(v')$ is independent of v', i.e. when ψ is a linear function. Since $\psi(-v) = \psi(v)$ we obtain $\psi(v) = cv$ for some constant $c\in\mathbb{C}$. Thus, changing $r(u,v)$ to $\exp(cuv)r(u,v)$ we can assume that $\psi = 0$. Finally making a substitution $v\mapsto -v$ and exchanging the first two components in the equation (5.2) we get (taking into accound the unitarity condition) that

$$(\mathrm{pr}\otimes\mathrm{pr})(r_0(v)\phi^1 - \phi^2 r_0(v)) = (\mathrm{pr}\otimes\mathrm{pr})(-r_0(v)\phi^2 + \phi^1 r_0(v)),$$

or equivalently,

$$[\overline{r}(v), \phi^1 + \phi^2] = 0$$

which means that ϕ is an infinitesimal symmetry of $\overline{r}$. Hence, $\phi = 0$. □

Remarks. 1. We don't know whether for every unitary non-degenerate solution $\overline{r}(v)$ of the CYBE there exists a unitary solution of the AYBE of the form $\frac{1\otimes 1}{u} + r_0(v) + \ldots$ such that $(\mathrm{pr}\otimes\mathrm{pr})(r_0(v)) = \overline{r}(v)$.
2. In the case when $\overline{r}(v)$ has non-trivial infinitesimal symmetries the proof above shows that there are no more liftings of $\overline{r}(v)$ to a unitary solution $r(u,v)$ of the AYBE (considered up to rescaling) than infinitesimal symmetries of $\overline{r}(v)$. More precisely, such a lifting $r(u,v) = \frac{1\otimes 1}{u} + r_0(v) + \ldots$ is uniquely determined by $r_0(v)$ and the difference between r_0's for two liftings always has form $\phi^1 - \phi^2 + c\cdot 1\otimes 1$ for some infinitesimal symmetry ϕ and some constant c.

The above theorem can be applied in particular to the case when $\overline{r}(v)$ is an elliptic non-degenerate solution of the CYBE. Indeed, this follows from the following lemma (which I learned from Pavel Etingof).

Lemma 5.1. *Elliptic non-degenerate solutions of the CYBE have no infinitesimal symmetries.*

Proof. The idea is to look at residues of such a solution at poles. Let us denote $V = \mathbb{C}^n$. Using the Killing form on $\mathrm{sl}(V)$ we can identify $\mathrm{sl}(V)\otimes\mathrm{sl}(V)$ with endomorphisms of $\mathrm{sl}(V)$. Then the residues are operators corresponding to the action of the group $G = (\mathbb{Z}/n\mathbb{Z})^2$ on $\mathrm{sl}(V)$ induced by an irreducible projective representation ρ of G on V (see [2],5.1,5.2). Let us denote by $\mathrm{Ad}\,\rho$ the representation of G on $\mathrm{sl}(V)$. It suffices to prove that if $A\in\mathrm{SL}(V)$ is such that

$$\mathrm{Ad}(A)\circ\mathrm{Ad}\,\rho(g)\circ\mathrm{Ad}(A)^{-1} = \mathrm{Ad}\,\rho(g)$$

then $A^{n^2} = 1$. But this equation means that for every $g\in G$ we have

$$A\rho(g)A^{-1} = c\cdot\rho(g)$$

for some constant $c\in\mathbb{C}^*$. Considering the determinants we see that $c^n = 1$, hence,

$$A^n\rho(g)A^{-n} = \rho(g).$$

It follows that A^n is scalar. Since it belongs to $\mathrm{SL}(V)$ we conclude that $A^{n^2} = 1$. $\square$

The conclusion one can draw from the above lemma and from theorem 6 is that elliptic solutions of the AYBE constructed from triple Massey products on an elliptic curve can be uniquely reconstructed from the limiting elliptic solutions of the CYBE. As we have shown in [18] the A_∞-category of elliptic curve (or at least the "transversal" part of it) can be recovered from the usual category of vector bundles and from the triple Massey products of the type considered in section 1. Hence, in some sense the information about all higher products of the A_∞-structure on elliptic curve (considered up to homotopy) is encoded in elliptic solutions of the CYBE.

6. Appendix

In this appendix we prove two formulas for which we could not find references in the literature. Let $\zeta(x,\tau)$ be the Weierstrass zeta-function associated with the lattice $\mathbb{Z}+\mathbb{Z}\tau$. Let $\wp(x,\tau) = -\zeta'(x,\tau)$ be the corresponding $\wp$-function. For a pair of rational numbers (r_1,r_2) we denote

$$\zeta_{r_1,r_2}(x,\tau) = \zeta(x+r_1+r_2\tau,\tau) - r_1\eta_1(\tau) - r_2\eta_2(\tau), \tag{6.1}$$

where η_1,η_2 are quasi-periods corresponding to the basis $(1,\tau)$ (i.e. $\eta_1(\tau) = \zeta(x+1,\tau)-\zeta(x)$, $\eta_2(\tau) = \zeta(x+\tau)-\zeta(x)$). The first formula is

$$\zeta(dx,d\tau) = \frac{1}{d}\cdot\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\zeta_{\frac{i}{d},0}(x,\tau) + \frac{x}{d}\cdot\sum_{i\in(\mathbb{Z}/d\mathbb{Z})^*}\wp(\frac{i}{d},\tau). \tag{6.2}$$

For the proof let us fix τ and denote by $f(x)$ the difference between the LHS and the RHS. Then one immediately checks that $f(x)$ is holomorphic on the entire plane, $f'(x)$ is doubly periodic with respect to the lattice $\mathbb{Z}+\mathbb{Z}\tau$, and $f(-x) = -f(x)$. Therefore, $f(x) = c\cdot x$ for some constant c. Hence, it suffices to check the identity obtained from (6.2) by differentiation:

$$\wp(dx,d\tau) = \frac{1}{d^2}\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\wp(x+\frac{i}{d},\tau) - \frac{1}{d^2}\sum_{i\in(\mathbb{Z}/d\mathbb{Z})^*}\wp(\frac{i}{d},\tau).$$

But this can be proven directly from the definition of the $\wp$-function as a series.

As a corollary of (6.2) we immediately get that

$$\eta_2(d\tau) = \eta_2(\tau) + \frac{\tau}{d}\cdot\sum_{i\in(\mathbb{Z}/d\mathbb{Z})^*}\wp(\frac{i}{d},\tau).$$

Now it is easy to derive the following version of formula (6.2):

$$\zeta_{0,\frac{j}{d}}(dx,d\tau) = \frac{1}{d}\cdot\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\zeta_{\frac{i}{d},\frac{j}{d}}(x,\tau) + \frac{x}{d}\cdot\sum_{i\in(\mathbb{Z}/d\mathbb{Z})^*}\wp(\frac{i}{d},\tau). \tag{6.3}$$

The second formula makes a connection between the special values of the Kronecker function and Weierstrass zeta-function. Namely using the notation (2.2) we have

$$2\pi i F_{\frac{k}{d},\frac{l}{d}}(0,dx,d\tau) = \sum_{j\in\mathbb{Z}/d\mathbb{Z}}\exp(-2\pi i\frac{kj}{d})[\zeta_{\frac{j}{d},\frac{l}{d}}(x,\tau) - \zeta_{\frac{j}{d},0}(-\frac{k\tau}{d},\tau)]. \tag{6.4}$$

where d, k and l are integers, $d>0$, k is not divisible by d. The proof of this formula is straightforward. Indeed, changing x one can reduce to the case $l=0$. Then the difference between the LHS and the RHS is a holomorphic function of x, doubly periodic with respect to the lattice $\mathbb{Z}+\mathbb{Z}\tau$, vanishing at $x = -\frac{k\tau}{d}$, so it vanishes identically.

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